# ON NORMAL SUBGROUPS OF DIRECTED SQUARE COMPLEXES FROM DIAGRAM GROUP OVER DIRECT PRODUCT OF TWO FREE SEMIGROUP PRESENTATIONS

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**ABSTRACT**: To every semigroup presentation and every baseword, we associate a diagram group, defined as the fundamental group of the square complex. In this paper, we establish a new square complex over the direct product of two semigroup presentations. We first state and prove several theoretical properties of square complexes, including the number of vertices and edges, the maximum subtree, the generators and relations of the square complex. In addition, we provide formulas for the total number of generators of the diagram group and the number of normal subgroups of square complexes.

**KEYWORDS:** Diagram group, Fudamental group, Semigroup presentation.

## 1. Introduction

The first definition of diagram groups was introduced by Meakin and Sapir in (1997, 1999, 2002, 2006a, 2006b). However, their student, Kilibarda (1994; 1997) had worked out the first result on diagram group. Her work proved that every equivalence class of semigroup diagrams contains a unique diagram without dipoles. Such diagrams are called 'reduced'. Since then, many articles have been dedicated to the understanding of which groups can be described as (subgroups of) diagram groups and which properties can be deduced from such a description.
Given two semigroup presentations $P_1 = \langle \Sigma_1 \mid R_1 \rangle$ and $P_2 = \langle \Sigma_2 \mid R_2 \rangle$ it is accepetedly that their direct product $P_1 \times P_2$ has a presentation

$$\langle \Sigma_1, \Sigma_2 | R_1, R_1, ab = ba(a \in \Sigma_1, b \in \Sigma_2) \rangle$$

An immediate consequence of this is that $P_1 \times P_2$ finitely generated if and only if both $P_1$ and $P_2$ are finitely generated, and is finitely presented if and only if both $P_1$ and $P_2$ are finitely presented.
In their paper [2], Guba and Sapir explored several group-theoretical operations and demonstrated that the class of diagram groups is closed under these operations. These operations included finite direct products (a result initially due to Kilibarda [8]), free products, and a special operation (denoted by ·), which was used to construct an example of a diagram group that is finitely generated but not finitely presented.
In the construction described below, we consider a complex structure in conjunction with a graph of group structures. Specifically, we introduce a set $F$, which is disjoint from the vertex set $V$ and the edge set $E$. The set $F$ is called the set of cells. To each element in $F$, we assign a closed path, referred to as the defining path of the cell. In this context, we define a homotopy relation on the set of paths in the usual manner. Additionally, we can define the fundamental group of a complex $K$ with a basepoint www, denoted by $\pi_1(S(P), w)$.
In the construction described below we will have a complex structure on together with the graphs of groups structure. This means that we have a set $F$ which is disjoint from $V$ and $E$. This set is called the set of cells. We also have a mapping that assigns a closed path to each element in F. This path is called the defining path of the cell. Given a complex we define the homotopy relation on the set of paths in a standard way. Also, one can define the concept of the fundamental group of $K$ with basepoint $w$, we denote this group by $\pi_1(S(P), w)$.
We focus on complexes that have a graph of groups structure on their skeletons, and we refer to such structures as complexes of groups. The concept of a complex of groups is already well-established and widely used in the literature (see [6]). In our context, every square complex of groups, as we define it, is also a square complex of groups in the sense used in [6]; however, the converse is not necessarily true. In general, a square complex of groups is a structure that involves not only vertex groups $G_v$ , $v \in V$ and edge groups $G_e$ , $e \in E$., but also additional structure.

## 2 PRELIMINARIES

In this section, we introduce some concepts, terminologies, and theorems, such as words, semigroup presentation, graphs, and square complexes that are necessary to highlights.

**Definition 2.1** Let $X$ be a non empty set (alphabet). A word $W$ on $X$ is defined to be of the form $W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ such that $n \geq 0, x_i \in X, \varepsilon_i = \pm 1$. The length of $W$ is n and can be written as $L(W) = \sum_{i=1}^{n} |\varepsilon_i| = n$. The empty word is the word when n is zero, denoted by 1.

**Definition 2.2** Let $\Sigma$ be the set of alphabets. A semigroup presentation $P$ is a pair $P = (\Sigma \mid R)$, where $R \subseteq \Sigma \times \Sigma$. An element $x \in \Sigma$ is called a 'generating symbol'; while an element $(U, V) \in R$ is called 'defining relation', and is usually written as $U = V$. The semigroup defined by presentation is ${}^{\Sigma^+}\!/_{\approx}$, where $\approx$ is the smallest congruence on $\Sigma^+$ containing $R$. More generally, a semigroup $P$ is said to be defined by the presentation $(\Sigma \mid R)$, if $S \cong {}^{\Sigma^+}\!/_{\approx}$. Thus, elements of $S$ are in one-one correspondence with congruence classes of words from $\Sigma^+$ represents an element of $P$. For the sake of simplicity, it will be always assumed that the set of relations $R$ in every semigroup presentation $P = (\Sigma \mid R)$, satisfies the following condition: if $(U, V) \in R$, then $(V, U) \notin R$.

**Definition 2.3** A graph $\Gamma$ is consisting of five pairs $(E, V, i, \tau, -1)$ where $V$ and $E$ are two disjoint finite sets. Set $V$ is known as the set of vertices; while $E$ as the set of edges. Symbols $i, \tau, -1$ are functions:

$$i : E \longrightarrow V \quad , \quad \tau : E \longrightarrow V \quad , \quad -1 : E \longrightarrow V$$

such that:

$$i(e) = \tau(e^{-1}), \tau(e) = i(e^{-1}) \quad , e \neq e^{-1} \ \forall \, e \in E.$$

If $e$ is an edge, then $i(e)$ is called the 'initial vertex' of $e$, and $\tau(e)$ is called the 'terminal vertex' of $e$.

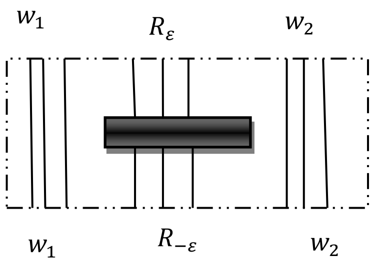
**Definition 2.4** Let $P = \langle \Sigma \mid R \rangle$ be a semigroup presentation, i.e. an alphabet $\Sigma$ and a collection of relations $R$ of the form $w_1 = w_2$ where $w_1, w_2$ are positive words written over $\Sigma$. In the sequel, we will always assume that $R$ does not contain obvious redundancy, i.e. if $w_1 = w_2$ is in $R$ we will assume that $w_2 = w_1$ is not in $R$. In particular, $R$ does not contains a relation of the form $w_1 = w_1$.

**Definition 2.5** Let $P = \langle \Sigma \mid R \rangle$ be a semigroup presentation. The square complex $S(P)$ is the complex whose vertices are the positive words written over $\Sigma$; whose edges $[w_1, w_1 = w_2, w_2]$ connect two words aub and $avb$ if one can be obtained from the other by applying a relation $w_1 = w_2$ from $R$.

**Definition 2.6** Let $P = \langle \Sigma \mid R \rangle$ be a semigroup presentation, where $\Sigma$ is a set of alphabet and elements of $R$ are in form of pairs of positive

words $R_\varepsilon = R_{-\varepsilon}$ on $\Sigma$. An atomic picture $A$ over $P$ is of the form shown in Figure 1.

$A = (w_1, R_\varepsilon \to R_{-\varepsilon}, w_2)$




$\leftarrow i(A) = w_1 R_\varepsilon w_2$

$\leftarrow \tau(A) = w_1 R_{-\varepsilon} w_2$

*Figure 1 Atomic picture A over P*

The inverse of $A$ is $A^{-1} = (w_1, R_{-\varepsilon} \to R_\varepsilon, w_2)$, such that:

$$i(A^{-1}) = \tau(A) = w_1 R_{-\varepsilon} w_2 \ \text{ and } \ \tau(A^{-1}) = i(A) = w_1 R_\varepsilon w_2.$$

Thus, we can relate A as an edge

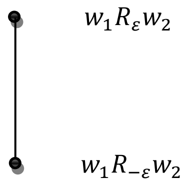



*Figure 2 An edge of square complex*

of a square complex.

Suppose that $A_1, A_2$ are two atomic picture such that $\tau(A_1) = i(A_2)$. Then we may define the composition $A_1 \circ A_2$ as a picture

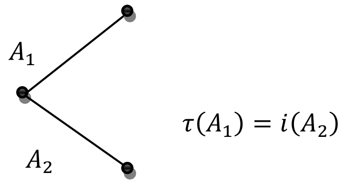



*Figure 3 The composition $A_1 \circ A_2$*

Collection of all possible composition of atomic pictures can be written as a connected square complex such that the edges are atomic pictures and vertices are all possible positive words on $\Sigma^+$. As a connected square complex, we may obtain the fundamental group, denoted by $D(P)$. This group is called the diagram group. (Refer[9])

**Definition 2.7** Let $P = \langle \Sigma \mid R \rangle$ be a semigroup presentation. A square complex graph $S(P)$ can be associated by the following procedure:

i. Consider as a 1-complex $S(P)$ coincide with the semigroup presentation $P$ viewed as a graph:
   a. The vertices are all positive words on $\Sigma^+$,
   b. Positive edges are triples $(w_1, R_\varepsilon \to R_{-\varepsilon}, w_2)$, where $w_1, w_2 \in \Sigma^+$ and $M \to N \in R$.
   c. Negative edges are triples $(w_1, R_{-\varepsilon} \to R_\varepsilon, w_2)$ where $w_1, w_2 \in \Sigma^+$ and $N \to M \in R$.
   d. If $e = (w_1, R_\varepsilon \to R_{-\varepsilon}, w_2)$ is an edge of $S(P)$, then $e^{-1} = (w_1, R_{-\varepsilon} \to R_\varepsilon, w_2)$, $i(e) = w_1 R_\varepsilon w_2$ and $\tau(e) = w_1 R_{-\varepsilon} w_2$.
   e. Pictorially, see (Figure 4 and Figure 5):

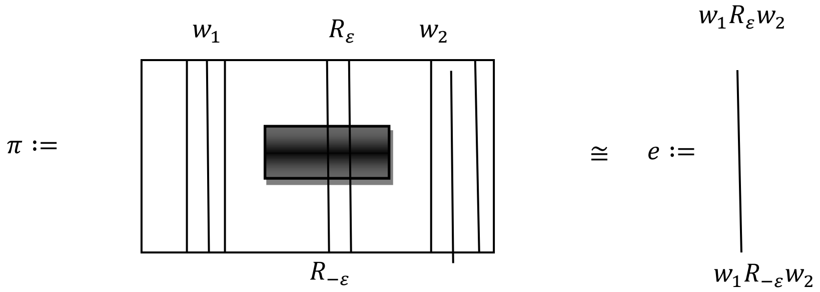



*Figure 4 $(w_1 R_\varepsilon w_2, w_1 R_{-\varepsilon} w_2)$-Picture*

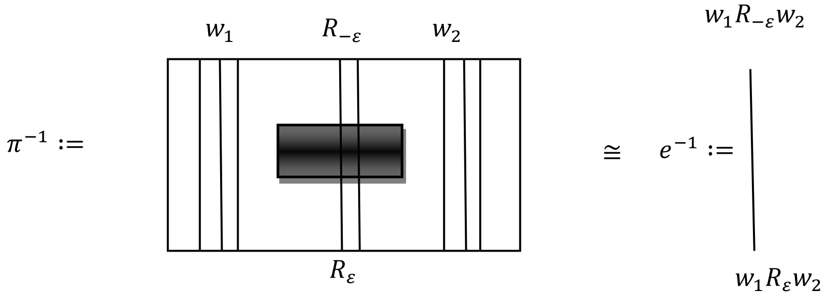



*Figure 5 $(w_1 R_{-\varepsilon} w_2, w_1 R_\varepsilon w_2)$-Picture*

ii. In a square complex the 2-cell of $S(P)$ are 5-tuples of the form $(w_1, M_1 \to N_1, w_2, M_2 \to N_2, w_3)$ where $w_1, w_2, w_3 \in \Sigma^+$ and $(M_i, N_i) \in R$ such that a 2-cell has the following defining path:

$$(w_1 M_1 w_2,\ M_2 \to N_2, w_3)(w_1, M_1 \to N_1, w_2 N_2 w_3)(w_1 N_1 w_2,\ M_2 \to N_2, w_3)^{-1}$$

From the previous figure, it is noticed that 2-cells correspond to independent applications of the relations from $R$. Applications of relations $M_1 \to N_1$ and $M_2 \to N_2$ to a word $w_1$ are called 'independent' if the corresponding occurrences of $M_1$ and $M_2$ do not have common letters: it does not matter which relation applies first and which relation applies second, it will be the same result. A note is also given to that diagram $\Delta$ corresponding to the path $(w_1, M_1 \to N_1, w_2 N_2 w_3)(w_1 N_1 w_2,\ M_2 \to N_2, w_3)$ that has been considered in the diagram corresponding to the defining path of the 2-cell $(w_1, M_1 \to N_1, w_2, M_2 \to N_2, w_3)$ is $\Delta o \Delta^{-1}$. Hence, this 2-cell is determined by the diagram $\Delta$.

Squier et al. (1994) introduced this complex. They introduced the 1-skeleton of the complex and a homotopy relation on the set of paths coinciding with the natural homotopy relation induced by $S(P)$.

**Definition 2.8** Let $P = \langle \Sigma \mid R \rangle$ be a semigroup presentation and $w \in \Sigma^+$ a baseword. The diagram group $D(P, w)$ is the fundamental group $\pi_1(S(P), w)$ of the Squier square complex $S(P)$ based at $w$.

**Theorem 2.9** [8]Let $P$ be a semigroup presentation, and $w \in \Sigma^+$ over the alphabet of $S$. Then, the diagram group $D(S(P), w)$ is isomorphic to the fundamental group $\pi_1(S(P), w)$ of the square complex $S(P)$ with basepoint $w$.
**Proof:** See Guba and Sapir (1997).
**Proposition 2.10** [2] Free products of diagram groups are diagram groups.
**Proof:** Let $I$ be a set. For every $i \in I$, let $P_i := \langle \Sigma_i \mid R_i \rangle$ be a semigroup presentation and $w_i \in \Sigma^+$ i a baseword. Without loss of generality, we assume that the alphabets$\Sigma_i$ are pairwise disjoint. Set

$$P = \langle \{w\} \cup_{i \in I} \Sigma \mid \cup_{i \in I} R_i \cup \{w = w_i, i \in I\} \rangle$$

Then $D(P, w)$ is isomorphic to the free product $\times_{i \in I} D(P_i, w_i)$ because $S(P, w)$ coincides with the disjoint union of the $S(P_i, w_i)$ together with the new vertex $w$ which is adjacent to all the $w_i$.

## 3. Materials and methods

**CONSTRUCTION OF SQUARE COMPLEX GRAPH FROM SEMIGROUP PRESENTATION**
In our previous research, we obtained the square complex from diagram group of semigroup presentations ${}^2S =< a\,, b\ : a = b\ >$ and ${}^3S =< x\,, y\,, z\ :\ x = y\,, y = z\,, z = x >$. See Kalthom M. and Ahmad A. [11],[12]. In this paper, the resercher will discuss the square complex from diagram groups of direct product of two free semigroups generated by $A, B, X, Y$ and $Z$. Thus the square complexes given by
$P = {}^2P \times {}^3P = \langle A\,, B, X\,, Y\,, Z\ | ax = ay, ay = az\,, az = ax, bx = by, by = bz\,, bz = bx(a \in A, b \in B, c \in C, x \in X, y \in Y, z \in Z) \rangle$ will be constructed.

Let $L(w) = 1$, and $w \in \Sigma^+$. So, the connected square complex graph is given by Figure 6

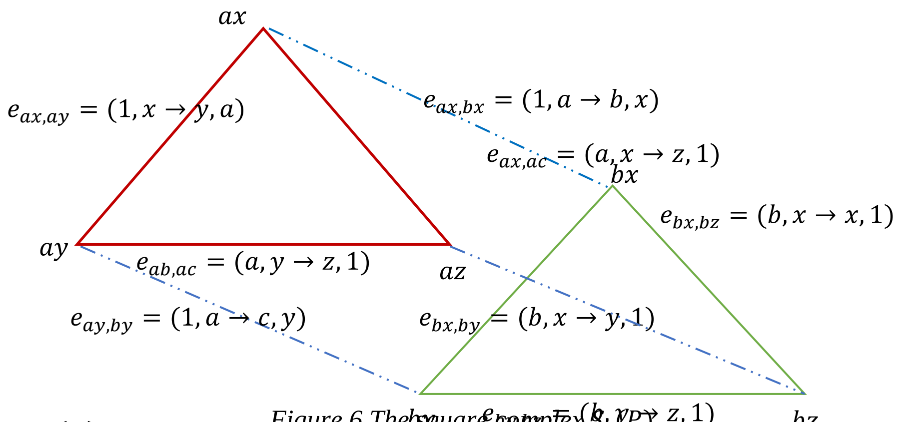



*Figure 6 The square complex $S_1(P)$*

Let $L(w) = 2$ is given by Figure 7.

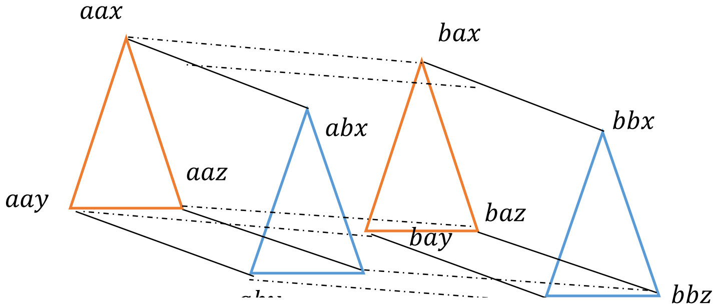



*Figure 7 The square complex $S_2(P)$*

Note that $S_2(P)$ are two copies of $S_1(P)$, and each two vertices are joined together in each copy, respectively. Likewise, in the case of two copies of $S_2(P)$, the square complex graph $S_3(P)$ may be obtained by repeating similar procedures to get $S_4(P)$ and so on.
From those diagrams, we can conclude some properties.

**Observation 3.1** Two vertices $w_1$ and $w_2$ are connected if and only if $L(w_1) = L(w_2)$.
**Observation3.2**If $L(w_1) = L(w_2)$. Then $\pi_1(S(P), w_1) \cong \pi_1(S(P), w_2)$.
**Observation 3.3** Vertices of $S_n(P)$ are all words of length $n$.
In this recent study, the researcher will expand square complex of diagram group.
**Theorem3.4** Let $P = \langle A, B, X, Y, Z \mid ax = ay, ay = az, az = ax, bx = by, by = bz, bz = bx(a \in A, b \in B, c \in C, x \in X, y \in Y, z \in Z)\rangle$ be a semigroup presentation and $w \in \Sigma^+$ in set $A \cup B \cup X \cup Y \cup Z$. Then the square complex of diagram group $D(S(P), w)$ contains $3(2^n)$ vertices and $e_n = 2e_{n-1} + 6(n-1)$ edges, and $e_1 = 9$.
**Proof.** By arguing the induction on $n$. For $n = 1$, the connected square complex graph of the diagram group $D(S_1(P), ax)$ contains $3(2^1) = 6$ vertices, and the total number of edges in $S_1(P)$ is $e_1 = 9$ see Figuare 6.
For $n = 2$, the square complex from diagram group $D(S_2(P), aax)$ contains $3(2^2) = 12$ vertices, and the total number of edges in $S_2(P)$ is $e_2 = 2e_{n-1} + 6(n-1) = 2e_{2-1} + 6(2-1) = 18 + 6 = 24$.
For $n = k$, assume $S_k(P)$ contains $3(2^k)$ vertices, and the number of edges is $3(2^{k+1})$.
For $n = k+1$, by the definition $S_{k+1}(P)$ is just two copies of $S_k(P)$, and since $S_k(P)$ contains $3(2^k)$ vertices, $3(2^{k+1})$ edges, then $S_{k+1}(P)$ contains $2[3(2^k)] = 3(2^{k+1})$ vertices, $2[3(2^{k+1})] = 3(2^{k+2})$ edges.
**Lemma 3.5** Let $P = \langle A, B, X, Y, Z \mid ax = ay, ay = az, az = ax, bx = by, by = bz, bz = bx(a \in A, b \in B, c \in C, x \in X, y \in Y, z \in Z)\rangle$ be a graphical presentation**.** The general formula for the total number of generators of diagram group in the square complex $S_n(P)$ is $e_n = 2e_{n-1} + 1$, where $(n = 1, 2, \dots, k)$ and $e_1 = 5$.
**Proof.** As defined above, $S_n(P)$ is two copies of $S_{n-1}(P)$, then the maximal subtree $T_n$ in $S_n(P)$ is two copies of the maximal subtree $T_{n-1}$ in $S_{n-1}(P)$. We know that the edge does not belong to the maximal subtree will be a generator. Thus, the generators of the fundamental group $\pi_1(T_n)$ is two copies of generators in $\pi_1(T_{n-1})$ plus one generator between two maximal subtrees. Hence the number of generators of $\pi_1(T_n)$ is $e_i = 2e_{i-1} + 1$.
**Lemma 3.6** The number of normal subgroup s of the square complexes $S_n(P)$ is $N_n = 2^n$.
**Proof.** $N_1 = 2^1 = 2$**.** Since there are two subgroups of $S_1(P)$, the formula works for $n = 1$. Then supposing that the formula holds for all natural numbers less than $n$, we will show that it holes for $n$. Since $S_n(P)$ is two copies of $S_{n-1}(P)$, and $S_{n-1}(P)$ contains $2^{n-1}$normal subgroups, then $N_n = 2(2^{n-1}) = 2^n$.

## 3. Conclusion

The paper provided, a new technique which has been explored to study diagram groups that was previously obtained from direct product of two semigroup presentations, we have determined the connected square complex graph $S_n(P)$ , n ∈ N form diagram groups that were obtained from direct product of two semigroup presentations using the methods of covering space theory.

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