Analyzing Some Musical Tones Using Fast Fourier Transform

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Publishing date: 9/1/2025

Abstract:
This paper presents the analysis of some musical tones by applying Fourier analysis. It is a crucial tool with wide use in several fields such as music science and signal processing. Fourier analysis, specifically; fast Fourier transform converts musical tones and sounds from the time domain to the frequency domain by determining the fundamental frequencies and amplitudes. The study focuses on simple and complex tones by examining audio recordings of the number tones on telephone keypad, and two different and distinct symphonies utilizing MatLab software program. Moreover, it emphasizes the practical application of fast Fourier transform in analyzing musical tones, and the strong connection between mathematics and music science.
Keywords: Fourier analysis, Fast Fourier transform, Frequency domain, Musical tones, Signal processing, Waveform.

Introduction

Fourier analysis is a mathematical technique that converts a function of time into a function of frequency. It is named after the French mathematician Jean-Baptiste Joseph Fourier (1786-1830). This mathematical tool has a widely applications in several fields, and it is an essential tool to understand and evaluate many concepts in physics, engineering, and most notably, music analysis. Fourier series provides a structured approach to managing periodic signals and are hence highly helpful in comprehending sound [1]. Many sound synthesis techniques, such as additive synthesis, are based on the Fourier series, which states that musicians can combine sinusoidal waves to create complex timbres from sinusoidal components [2]. The use of Fourier analysis in acoustics and music theory goes beyond simple tone analysis. It is essential to digital signal processing, which makes technologies like musical sound synthesis possible [3]. Fourier analysis is relevant to music because it can break down musical tones into their individual frequencies, providing a thorough understanding of the acoustic characteristics that give each musical instrument its own distinct sound [4]. In other words, Fourier Transform breaks down the complicated waveforms into more manageable parts, like sines and cosines, which are simpler to interpret and analyze [5] [6]. Because the sound waves in this representation can be "dissected" into different frequencies, a crucial step in analyzing musical tones, it can be used to analyze sound waves [7] [8]. Techniques such as fast Fourier transform have noticeably accelerated on how sound is processed in real-time with two or extra human beings performing or musicians alone, and greater specifically sound engineers. Instead, has it supplied greater possibilities in the sound diagram and tone manufacturing [9]. Fourier transform is a powerful lens through which to view the subtleties of music. Technology advancements will open up a world of new possibilities for sound engineering and music research and applications [10]. Fourier approuch lays the groundwork for future study of analyzing musical tones, sounds, and also advancement in the fields of digital audio technology, signal processsing, and music theory by using Fourier transformations [14].

This research paper will test musical tones, with a particular emphasis on recordings made of some simple tones and different symphonies. The essential aim of this paper is to analyze and contrast the fundamental frequencies of the chosen musical tones using the algorithm of fast Fourier transform by MatLab software.

1. Continuous Fourier Transform[1]

The Fourier transform, a computational tool, with a wide usage in several fields of science such as a mathematical tool to alter the problem into one that can be greater without difficulty solved. Fourier transform breaks down a waveform into sinusoids of unique frequency which sum to the authentic waveform. It identifies the exceptional frequency sinusoids and their respective amplitudes. The formula of Fourier transform is an integral form described as:

$$\mathcal{F}\big(f(x)\big) = \int_{-\infty}^{+\infty} f(x)e^{-i2\pi xs}dx$$

The Fourier transforms of a function of x gives a function of s, where s is the wave number.

$$f(x) = \int_{-\infty}^{+\infty} F(s)e^{ixs}dx \Leftrightarrow F(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-ixs}dx$$

The Fourier transform of a function of t gives a function of ω where ω is the angular frequency:

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

1.1 The Convolution Theorem [16]

One of the mathematical tools is the convolution operation on two continuous functions. The convolution of Fourier transform is generally defined as the integral of the product of the two functions when one of them reversed and shifted.

$$(f*g)(t)=\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

Where the symbol" $*$" refers to the convolution operation.

In specific, the convolution theorem asserts that the pointwise product of the Fourier transforms of the signals $f(t)$ and $g(t)$ is equal to the Fourier transform of the convolution of $f(t)$ and $g(t)$, $(f*g)(\tau)$.

$$F(\omega)\ F(\omega)=\mathcal{F}[(f*g)(\tau)]$$

Furthermore, the convolution is equal to correlation if one of the functions is symmetrical.

1.2 The Parseval's Theorem[12]

The theorem of Parseval shows the exact power of a signal that described with the function of $h(t)$ is similar or not whether it is computed in signal space or frequency space:

$$\int_{-\infty}^{+\infty} h^2(t)dt=\int_{-\infty}^{+\infty} |H(f)|^2 df$$

So, the power spectrum, $P(f)$, is shown as

$$P(f)=|H(f)|^2\ ,-\infty<f<+\infty$$

2. Complex Form of Fourier Transform[11]

For a real function $f(t)$, the Fourier transform will usually not be real. Infact, the imaginary part of the Fourier transform of a real function is

$$Im[F(s)]=\frac{F(s)-F(s)^*}{2i}=\frac{1}{2i2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\,e^{-isx}dx-\int_{-\infty}^{\infty} f(x)\,e^{isx}dx\right]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,si\,n(sx)\,dx=F_{sin}(s)$$

The Fourier transform will have both real and imaginary parts

$$F(s)=F_{cos}(s)+\ iF_{sin}(s)$$

Where $F_{sin}(s)$ is the Fourier sine transform, $F_{cos}(s)$ the Fourier cosine transform.

One hardly ever uses Fourier sine and cosine transforms. We practically talk about the complex Fourier transform.

3. Discrete Fourier Transform(DFT)[12]

In general, the Discrete Fourier transform, abbreviated as DFT, is considered as the process of transforming a function or signal from time space to a sample space separated with respect to time T. Therefore, it is called the discrete Fourier transform of N points described as a sequence

$$f[0],f[1],\ldots,f[k],\ldots,f[N-1]$$

The Fourier Transform of a the signal $f(t)$ would be

$$F(j\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt$$

Since the integrand existed only at the sample point k we have

$$F(j\omega)=\int_{0}^{(N-1)T} f(t)e^{-j\omega t}\,dt$$

Or

$$F(j\omega)=\sum_{k=0}^{(N-1)} f[t]e^{-j\omega kt}$$

The discrete Fourier Transform works with the data as if it was periodic and evaluate signal for the fundamental frequency and its components.

4. Fast Fourier Transform(FFT)[12]

The fast Fourier transform (commonly abbreviated as FFT) is a highly efficient algorithm developed in the 1960s to calculate the discrete Fourier transform of a given signal very quickly by reducing the number of operations required by hundreds of times. With the discrete Fourier transform, this quantity is immediately associated to toN^2 where N is the length of the transform.

$$F[n] = \sum_{k=0}^{(N-1)} f[k]W_N^{nk}\ , W_N^{nk} = e^{-j\frac{2\pi}{N}nk}.$$

For most problems, N is chosen to be at least 256. That is in order to get a suitable approximation for the sequence under consideration.

4.1 Presenting Speed of FFT[13]

In fact, the DFT needs $N^2$of complex multiplications. Every stage of FFT, $\frac{N}{2}$ complex multiplications is needed to combine the results of the preceding stage. Sincethere are $(log_2 N)$ stages, the number of complex multiplications needed to evaluate N-point DFT with the FFT is about$\frac{N}{2}$ $(log_2 N)$.

Table 2: Speed of Different stages

N	N^2 (DFT)	$\frac{N}{2}$ $(log_2 N)$ (FFT)	*Saving*
32	1024	80	92%
256	65536	1024	98%
1024	1048576	5120	99.5%
2048	4194304	11264	99.7%
16384	268435456	114688	99.95%
131072	17179869184	1114112	99.99%

5. Testing Tones in Time-Frequency Domain [14]

The retrieval of music is greatly improved when such analysis can be efficiently performed. Fourier analysis gives an inadequate description of musical sounds, in view of the conjecture that the ear is sensitive only to sinusoidal waveforms. In particular, Fourier tool, where any continuous function can be approximated by a sum of simple sine or cosine functions is one of the great results in mathematicsandphysics science. The strategy of Fourier analysis is significantly used inmany functions involving vibrations periodic phenomena, and different various contexts. Musical tones can be viewed to be periodic, and they can be generated by means of the superpositionof associated frequency components, harmonically.

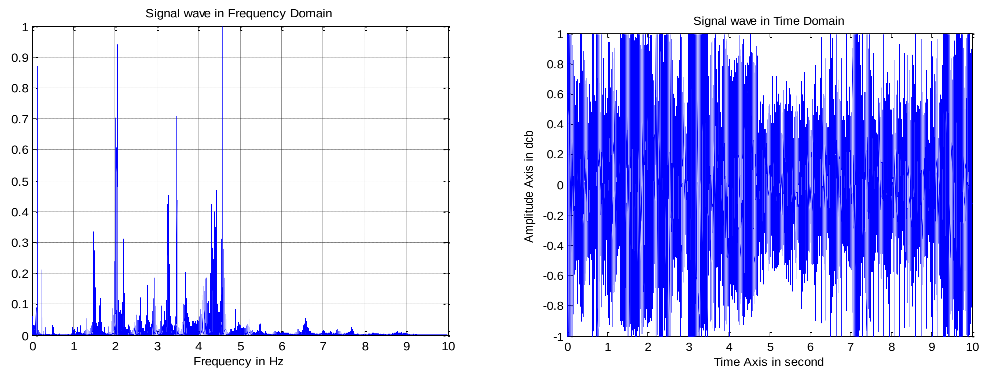

Fig1: Time-Frequency domain of Signal wave

Figure1 expresses the amplitudes of the wave at each moment in the time domain. In contrast, the wave consists of different frequencies in the frequency domain.

6. Dual Tone Multi-Frequency (DTMF)

A DTMF signal consists of the sum of two tones with frequncies taken from two mutually exclusive groups. These frequncies were chosen to prevent any harmonics from being incorrectly detected by the receiver as some other DTMF frequency. Each pair of tones contains one frequency of the law group $(697Hz, 770Hz, 852Hz, 941Hz)$ and one frequency of the high group $(1209Hz, 1336, 1477Hz)$ and represents a unique symbol.

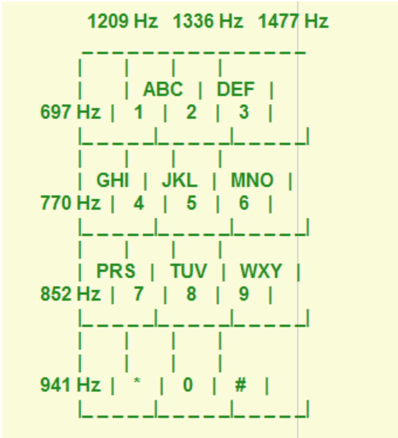

Fig2: Dual frequency of telephone keypad numbers and symbols

The figure 2 shows the multi frequencies allocated to the push-buttons of the telephone pad.

7. Technical Process of FFT Algorithm

MatLab program contains different mathematical applications, including the fast Fourier transform. It is an advanced discrete Fourier transform algorithm that analyzes waves at very high speed. Therefore, applying FFT code via MatLab to analyze any sound or musical tones takes basic steps to be sitting:

1. Time of the recording.
2. (recObj) function: Audio recording object function to input (receive) a sound signal.
3. (fftdemo) function: Fast Fourier transform function to demonstrating the output wave forms.

7.1 Case Study Using MatLab

In this section, different structure of tones will be taken to be examined using the algorithm of FFT as following:

1. Simple Tones

Obviously, mobile phones are widely used. When touching the number pad, a simple and very short musical sound is emitted from each number or symbol, which differs from the other. These numbers and symbols have charactaristic features about the wave form, frequencies, and amplitudes.

The amplitude of a wave is defined as the measurement of the shift of the wave from its rest position (is the measurement of the strength, intensity, or loudness of the wave), so it is basically calculated by looking on the graph of the wave, and measuring the exact height from the balance line. [15]. It is commonly noted with a sympol A.

$$A = |x|_{max} \;, \qquad where\ x\ is\ the\ shift.$$

Arithmeticaly, the sinusoidal wave equation is given by the formula:

$$y(t) = Asin\,(\omega t + \phi)$$

Where A is the amplitude, ω is the angular frequency, and ϕ is the phase constant.

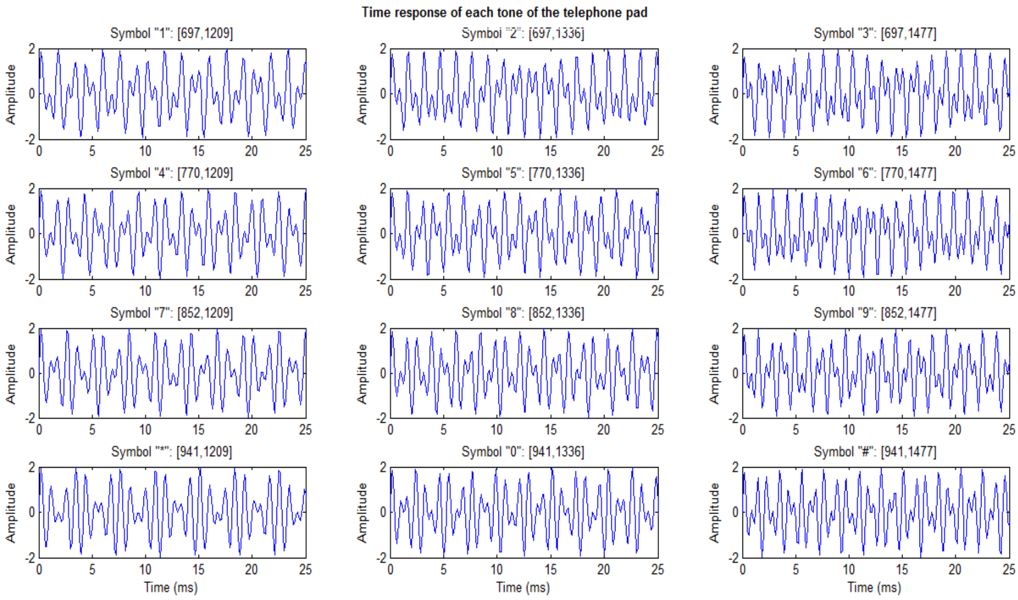

Fig3: Wave forms of the number pad and symbols in the time domain

As a certain case, tones of the symbols (1) and (2) in telephone pad are chosen to be explained as following:

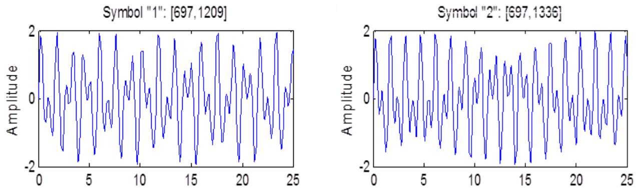

Fig4: Wave form of the tones (number 1 and 2) of the telephone pad in time dmain

So, the figure above explains that the two waves are simple, regular and periodic (sinusoidals). Therefore, the amplitude of the above tones is $A = 2cm$.

Now, after processing by FFT, we get

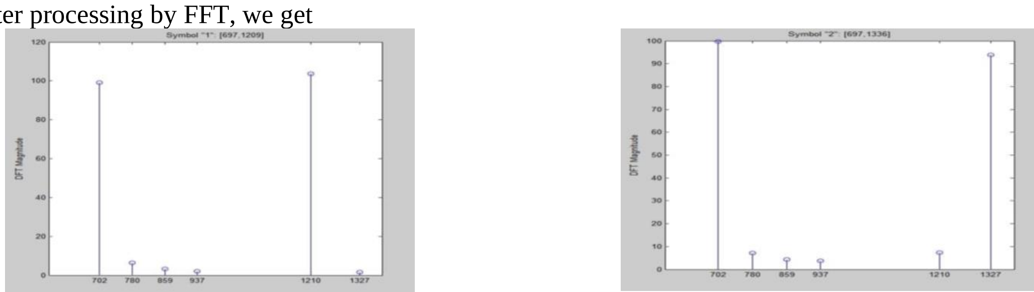

Fig5: Tones of (number 1 and 2) of the telephone keypad in frequency domain

Based on the figure 5, all frequencies of both number tones (1 and 2) are clear, where the exact frequencies are pointed with the long lines.

2. Complex Tones

It is really often to hear song or symphonies of famous musicians from different countries of the world. In this case, as complex tones, awonderful calm symphonies are chosen to be analyzed in both of time domain (for 10 seconds) and frequency domain as following:

i). Birds of Winter Symphony (Toyor Alshitaa)

This symphony was played by the famous Romanian musician, George Zamfir. Born in 1941 and known as "The Master of Pan Flute".

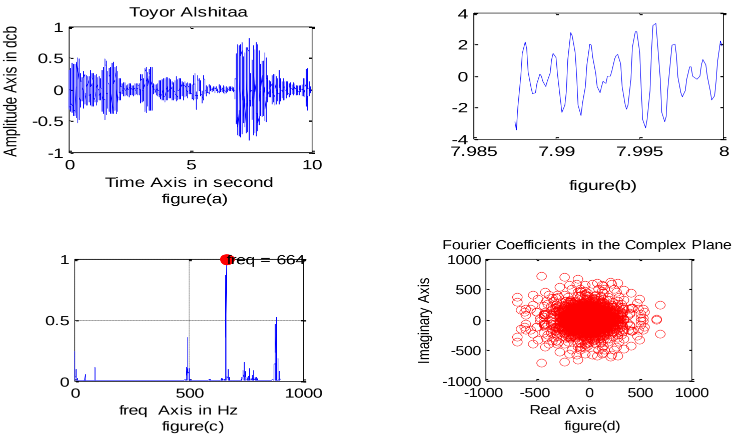

Fig6: Analyzing apart of the symphony (i) using FFT

ii). Moonlight Symphony (DhawAlqamar)

This symphony was played by the famous German composer, Ludwig Van Beethoven (1770-1827) using Piano instrument.

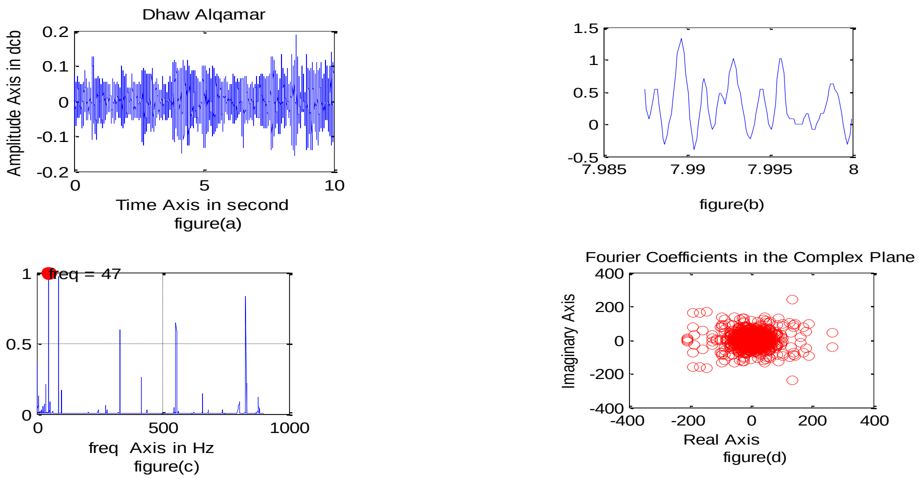

Fig7: Analyzing apart of the symphony (ii) using FFT.

From the above figures, it can be noticed that the wave motions of the tones are not periodic. Loudness and calmness of the tones have a direct effect on the power of frequency. Also, it should be considered that, as long as the part of symphony is longer in time, the higher frequencies of some tones would be appears, absolutely. The table below discussed the results in details:

Table3: Explaining the result description of the figures 6 and 7

Figures	**Birds of winter (ToyorAlshitaa) Symphony**	**Moonlight (DhawAlqamar) Symphony**
(a)	The wave shape of the tone after a time$T = 10\ s$. The loudness of the wave $is\ \ 1\ dcb$.	The wave shape after a time $T\ = 10s$. The loudness of the wave $is\ \ 0.2\ dcb$.
	Does not represent a simple harmonic motion.	
(b)	Represents the wave in small parts of a second to be more clearly	
(c)	Frequencies of the tone oscillate $0 \leq frequencey < 1000$.	
	The highest frequency of this tone is $664Hz$.	The highest frequency of this tone is $47Hz$.
(d)	Shows the Fourier coefficients in complex space. The $(x - Axis)$and $(y - Axis)$ represent the amplitudes a_k, b_k in order for each $k \in \mathbb{N}$.	
	The scattered spots represent **high frequencies** which amplitudesapproximately satisfy the inequalities:	
Coefficients a_k,b_k	$(-1000 \leq a_k, b_k < -500)$ Or $(-500 < a_k, b_k \leq 1000)$	$(-400 \leq a_k, b_k < -100)$ Or $(100 < a_k, b_k \leq 400)$
	The red spot represents all **small frequencies** which amplitudes approximately satisfy the inequalities:	
	$(-500 \leq b_k \leq 500)$, $(-500 \leq a_k \leq 500)$	$(-100 \leq b_k \leq 100)$, $(-100 \leq a_k \leq 100)$

The findings underscore the transformative power of mathematical analysis in understanding and appreciating the complex beauty of musical sounds.

8. Conclusion

The applications of Fourier concepts explore the strong mathematical deepness in music science, physics and signal processing. It was concluded that how very powerful and helpful the algorithm of fast Fourier transform in understanding the nature structure of musical tones. That is by breaking down any waves to their frequencies and amplitudes. This analytical journey is underpinned by mathematical rigor and assisted by MatLab software. It has illuminated the complex interplay between essential frequencies, harmonics, and the musical tones resultant of the symphonies. All of this was done by segmenting tone into its constituent frequencies.

9. Recommendations

Forthe interested researchers of this field, thefollowing suggestions are recommended for future study:

1. Exploring advanced noise reduction techniques in FFT analysis.
2. Comparing FFT results with other signal processing methods or software.
3. Investigating the use of FFT in other domains, such as image processing or bioacoustics.

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