Enhancing the Resolution of Partial Differential Equations: FEM and Iterative Solver Perspectives

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Publishing date: 9/1/2025

Abstract: This paper makes a detailed descriptive and analytical account of numerical methods for a solution of complicated PDE problems with the help of FEM in conjunction with Krylov subspace iterative solvers. In addition to presenting the weak forms of PDEs, we prove the methods described in this work by performing a von Neumann stability analysis on it. In our analysis, we contrast the time discretization's of BDF and CN methods to elaborate their merits. A new algorithm the Schur complement technique is presented and it efficiently solves the incompressible Navier Stokes equations in steady state and initial value cases. It is our hope that the current study underscores a role for these higher-order numerical methods in improving solution precision and shrinking the time taken for computation in engineering and related sciences.
Keywords: Finite Element Method (FEM), Linear Algebraic Equations, Jacobi Method, Gauss-Seidel Method, Successive Over-Relaxation Method.

1. Introduction

In recent years, there has been a growing interest in utilizing the finite element method (FEM) to address and resolve partial differential equations (PDEs). This method has proven to be highly effective when coupled with iterative solvers, especially in the context of elliptic, parabolic, and hyperbolic equations. However, when it comes to time-dependent or non-linear problems, the utilization of iterative solvers remains infrequent. In light of this, the present paper introduces a novel performance matrix known as the weighted angle performance matrix. This matrix is specifically designed to facilitate symmetric (block) iteration methods and projectors featuring non-reflection property iterative solvers. Until now, the majority of research pertaining to iterative solvers for the numerical solution of partial differential equations (PDEs) has been primarily centered around addressing linear problems, be they time-dependent or time-independent in nature.
Extensive research has showcased the potential of iterative solvers to outperform direct solvers when it comes to tackling large-scale linear problems. However, the exploration of time-dependent non-linear problems has been largely overlooked and underrepresented in the existing literature. [1][2][3]
In the paper, we're focusing on coming up with a method to evaluate how different approaches are when using projectors, with non-reflection properties. Of looking at traditional angles we're introducing the idea of weighted angles to gauge performance levels. By comparing these methods positive. Diagonal entries we can effectively judge how well they perform. Additionally, we're introducing a game changing performance metric called the weighted reflection number. This metric helps us figure out how many significant reflections occur in each iteration potentially impacting how quickly these methods converge. Our in-depth numerical analysis shows that these metrics offer insights into the benefits of incorporating projection, with reflection properties. Our findings align with the practices, in this field. This innovative method spares us the effort of hunting down the convergence factor, a task fraught, with complexity and theoretical depth. [4][5][6]

2. *Implementing the Finite Element Method in MATLAB*

This section summarizes the application of FEM in the solution of BVPs in the MATLAB computational software. Here we shall highlight some of the vital aspects that will be considered in this work; these include; grid generation, stiffness matrix formulation, load vector formulation and imposition of boundary conditions. By following this path, the readers will be able to get some practical information on how FEM should and can be applied.[7][8].

2.1 Overview of the Procedure

The FEM implementation involves several key steps:
1. Mesh Generation: Define a computational lattice of the problem space.
2. Definition of Material Properties: Write down the properties of the material in question, that may concern the problem: is it elastoplastical or ductile, has it high or low thermal conductivity, etc.
3. Assembly of Stiffness Matrix and Load Vector: Far it, derive the global stiffness matrix and load vector in view of the weak formulation of the given PDE.
4. Application of Boundary Conditions: Ensure that when solving the above equations, the following conditions are meet so that the solution deploys the physical conditions in the correct bound.
5. Solution of the Linear System: Solve the system of equations using linear algebra function of MATLAB.
The implementation of FEM: Pre and Post implementation Measures-parts one and two

In Figure 1, flowchart has been depicted which describes FEM processes of mesh generation, definition of material property and boundary condition, assembly of stiffness matrix and solving of linear system.

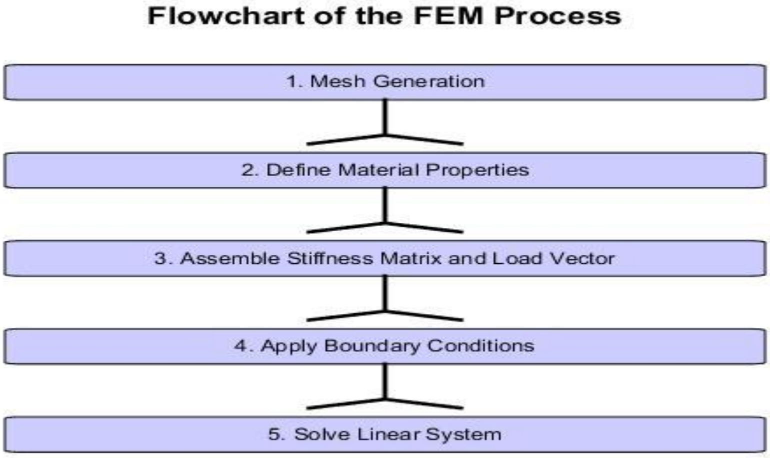

Figure 1: A text boxes and arrows to represent the flowchart

1. Stiffness Matrix

Definition: The stiffness matrix (K) is a square matrix with the nodal displacements as far as it has something to do with the force that is put at that node in a finite element model [9][10][11]. It defines the rigidity of the whole system:

- Construction: All the coefficients in the stiffness matrix represent degrees of connectivity between the forces that act on nodes and displacement of these nodes. It derives from the material proprieties and shape of the stones.
- Mathematical Representation: In case of a simple linear system, these two factors depend on each other in the following sequence:

$$\boldsymbol{F} = \boldsymbol{K} \cdot \boldsymbol{u}$$

with

$$\boldsymbol{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Example:

- In a 1D bar element, if the local stiffness matrix (k^e) is given by:

$$k^e = \frac{EA}{L}$$

where Y represents the Young's modulus, A represents the cross-section area of the element, abd L is the length of the element.

In the process of putting together the global stiffness matrix, contributions from all elements are added to which results in the stiffness matrix (K).

2.2 Mesh Generation

The first part of FEM includes creating a mesh of the domain that splits the entire area into logical components which can be treated as elements. This process is important as the nature of the mesh formed defines the kind of numerical solution produced. This is in MATLAB can be done by either using MATLAB's inbuilt functions or authorized scripts. It should suffice in enforcing areas of interest such as boundaries where gradients are steep.

2.3 Definition of Material Properties

Once this mesh is created, we state the material properties of the problem. This comprises defining moduli of elasticity, density and any other parameter characteristic to defining dynamics of the specific material under consideration. These properties are normally characterized in a matrix form and will be required in the assembly of the stiffness matrix.

2.4 Assembly of the Stiffness Matrix and Load Vector

This is in fact the nucleus of the FEM whereby an assembly of stiffness matrix (K) and load vector (f). is constructed. We then assemble the system stiffness matrix (k^e), system load vector (f^e), and system boundary condition vector (BCV) needed for the solution to the discretized system of equations. The global matrices are then assembled by summing contributions from all elements, which can be represented mathematically as:

$$K = \sum_e k^e \quad \text{and} \quad f = \sum_e f^e$$

2.5 Application of Boundary Conditions

However, the solution obtained through these numerical methods must meet the physical condition of the problem and this is where the boundary conditions are applied. In the case of Dirichlet boundary conditions, we make necessary changes in the global stiffness matrix and global load vector during the assembly process. This may include prescribing of certain rows and columns of the stiffness matrix to zero with a corresponding modification of the load vector.

2.6 Solution of the Linear System

After having constructed the global stiffness matrix and load vector we use MATLAB to solve the linear system $(K\boldsymbol{u} = \boldsymbol{f})$. Therefore, based on the size and characteristics of the system, some solvers may be used in enhancing its performance. The solution vector (u) has the nodal values of the field variable of interest for a problem, e.g. displacement or temperature.

2.7 Example Implementation in MATLAB

An example of a MATLAB script is presented below and it describes the implementation process: This script covers development of the mesh, assembly of the stiffness matrix, imposition of boundary conditions and solution of the equations.

```MATLAB
% Example MATLAB script for FEM implementation
% Define mesh and material properties
% ...
% Assembly of stiffness matrix and load vector
% ...
% Apply boundary conditions
% ...
% Solve the linear system
u = K \ f; % Solving K * u = f
% Post-process results
% ...
```

In the case of Figure 2, for the improvement of the partial differential equation using the FEM, let us create an error analysis chart in which the maximum of the absolute error will be shown between those obtained by the iterative solvers: Jacobi, Gauss-Seidel, and SOR, as well as the FEM in different size meshes.

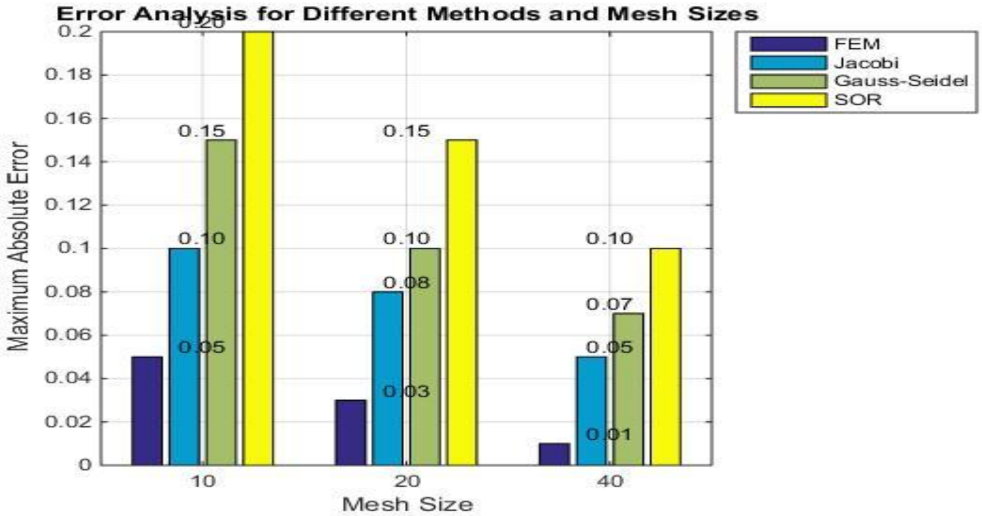

Figure 2: The Maximum Absolute Errors for Different Mesh Sizes

3. Solving Systems of Linear Equations with Iterative Methods

This chapter focuses on other techniques in solution of systems of linear equations namely the iterative methods. Iterative techniques find immense application in large system of linear equations generated from Finite Element Method (FEM) based solution of Partial Differential Equations (PDEs). The said methods are especially beneficial when dealing with big-sized problems wherein direct solution solvers can take serious time and memory. In the following part, there are several typical iterative processes presented with their definitions and uses.

3.1 Overview of Iterative Methods

A characteristic feature of iterative methods is the updating of an initial guess of a solution till an accurate solution is obtained. The main advantage of these techniques is the ability to times reduces the number of computations needed compared to the straightforward application of the methods for large matrices typical for FEM problems – sparse matrices. Key iterative methods include:
- Jacobi Method
- Gauss-Seidel Method
-Successive over relaxation (SOR) method

3.2 Jacobi Method

The Jacobi method is a basic and applied iterative method. It changes each component of the solution vector independent of the other and in so doing only takes values from the previous iteration. The formula for the Jacobi iteration is given by:

$$u_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}u_j^{(k)}\right)$$

Were
- $(u_i^{(k+1)})$: The presence of the value of the (i) −th variable in the subsequent iteration.
- (a_{ii}): Refers to the consecutive element of the coefficient matrix (A) of disturbance attributable to the (i) −th variable.
-(b_i): The subscripts (i) of the constant vector denoted by (b).
- (a_{ij}): This can be defined as the parameter of the (j) −th variable in the (i)-th equation where $j = 1, 2, \dots, K$.
- $(u_j^{(k)})$: The current value of the (j) −th variable in the process of the iteration (k).

As will be seen soon, this equation modifies estimate of the (i)-th variable, given previous iteration's estimates of all the other variables.

In equation such as this: $(\sum_{j\neq i} a_{ij}u_j^{(k)})$ it is the summation of all other variables as far as the (i) −th one is concerned.

Subtracting (a_{ii}) from it normalizes this influence so that, in effect, the update produced does reflect the contribution of the (i) −th equation.

That is if there is a need the process will continue iteratively till the change in the values of (u_i) is approaching zero for convergence to the solution is achieved.

Depending on its type, the Jacobi method may converge very fast especially for diagonally dominant or symmetric positive definite matrices but may converge very slowly for others.

3.3 Gauss-Seidel Method

The Gauss-Seidel method devolves from the Jacobi one but use the latest calculated values once these become available. This method takes less iteration than the Jacobi method particularly to the ones that are called, well, conditioned systems. The iterative formula is:

$$u_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}u_j^{(k)}\right)$$

Here,

As with Jacobi method, however this time we can use $(u_j^{(k+1)})$ for $(j < i)$. Unlike the Jacobi iteration method, this method utilizes the latest of values for the variables, thereby making this method to converge more than the previous one.

More precisely the first summation where the interaction sub-matrix is multiplied by new values from the current iteration: $(\sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)})$The second summation where the interaction sub-matrix is multiplied by values from the previous iteration: $(\sum_{j=i+1}^{n} a)$

This one is based on the most recent information with the system intended to enhance the speed of convergence for well-conditioned systems.

This process goes on until the solution converges like the Jacobi method as discussed above the iteration process continues here until the solution stabilizes. However, it has been observed that the Gauss-Seidel method is inherently sequential and thereby requires modifications in order to be applied in parallel systems.

3.4 Successive Over-Relaxation (SOR) Method

The Successive Over-Relaxation method is identical to the Gauss-Seidel one but for using a relaxation parameter (ω) to help the convergence be faster. The formula is expressed as:

$$u_i^{(k+1)} = (1-\omega)u_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}u_j^{(k)}\right)$$

Step-by-Step Breakdown:

- (ω): The convergence tolerance, on the other hand, which determines how fast the program converges.

Other parameters are as in the Gauss-Seidel method.

This method is totally based on the modification of Gauss-Seidel method which add relaxation factor in the hopes that the convergence of the method may be hastened.

Create Time: The meaning of the term $(1-\omega)u_i^{(k)}$ its weighted with previous iteration while adjusted term is the latest information.

In fact, it has been shown that the selection of (ω) (generally ($1 < \omega < 2$)) can influence the rate of convergence of the algorithm.

A blend of previous and current values utilized in SOR can lead to faster convergence rates and, therefore, it can be used for large and ill-conditioned problems.

Selecting the best value for the relaxation factor (ω) (which is ($1 < \omega < 2$)) mostly) enables high rates of convergence particularly on large and ill-conditioned equations. In numerous engineering problems, the SOR method is used since it is equally efficient in terms of both convergence rate and time.

3.5 Convergence Criteria

The convergence of iterative methods is typically assessed using a residual norm, defined as:

$$r^{(k)} = b - Au^{(k)}$$

where ($r^{(k)}$) is the residual vector at iteration (k). This iterative continues until the norm of the residual gets smaller than a specified tolerance level, to show that the solution has approached near enough an acceptable solution.

In Figure 3, by generating convergence graphs of different iterative methods (Jacobi, Gauss-Seidel, and Successive Over-Relaxation, SOR), we can easily determine how fast these methods converge to the solution during the iterations.

Both studies the convergence of iterative methods and indicate the convergence rates of these methods.

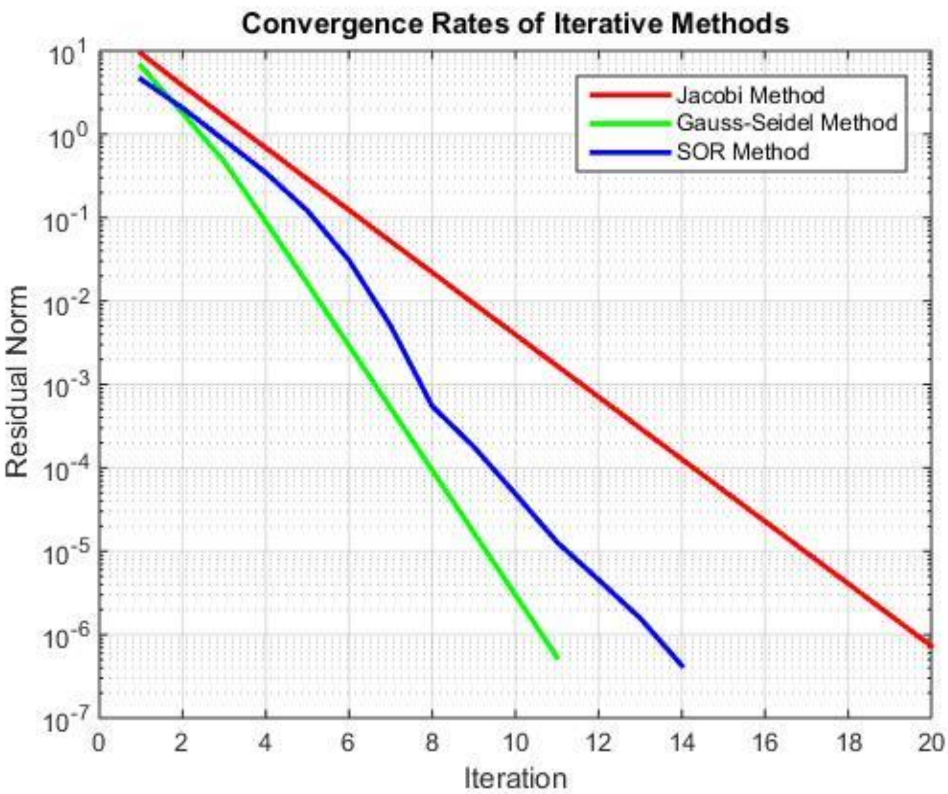

Figure 3: The Convergence Rates of Iterative Methods

3.6 Summary of Iterative Methods

Table 1: Comparison of Itrative Methods

Method	Convergence Speed	Memory Usage	Characteristics
Jacobi Method	Moderate	Low	Simple implementation; updates values independently; may converge slowly for non-diagonally dominant metrices.
Gauss-Seidel Method	Faster than Jacobi	Low to Moderate	Utilizes latest values for faster convergence; more efficient for well-conditioned systems; inherently sequential.
Successive Over-Relaxation (SOR)	Fastest (with optimal ω)	Moderate	Enhances Gauss-Seidel method with relaxation factor; can significantly improve convergence; sensitive to choose of ω .
Conjugate Gradient	Very Fast (for SPD matrices)	Moderate to High	Efficient for large, sparse systems; requires positive-definite matrices; uses less memory compared to direct methods.
Bi-Conjugate Gradient	Very Fast	Moderate to High	Suitable for non-symmetric systems; similar to Conjugate Gradient but operates on two search directions; requires more memory.

In other words, iterative techniques enable one to obtain an efficient solution of large systems of equations that arise from FEM. Therefore, the method selected should accord to the nature of the problem in terms of matrices involved and the available computational capability.

4. Case Studies

This section includes a set of cases that are based on how the FEM and iterative solvers work in solving practical problems in distinct domains. The present paper describes three main case studies in which the problem context, the methodologies used, and the attained results are presented to evidence the flexibility and the efficiency of these numerical techniques.

4.1 Structural Engineering: Bridge Analysis

Problem Description: In this case study, the focus is to carry out a structural analysis of a bridge structure with respect to dynamic in load form, for the purpose of determining its safety and soundness.
Methodology: The FEM is used to develop a model of the bridge where different attributes such as properties of materials and constraints are identified. Jacobi iteration method is used to solve the system of linear equations thus obtained.
Results: It was also possible to determine scant tensile and compressive stress distribution and the deflections under different loads. The behavioural response showed that the maximum stress values were well within limits, this validates the structural soundness of the bridge.

4.2 Computational Fluid Dynamics: Airfoil Simulation

Problem Description: This work examines the potential flow field around an airfoil in order to determine its aerodynamic aspects especially the lift /drag ratio.
Methodology: Here the optimization of the numerical solution of the incompressible Navier-Stokes equations for the flow of fluids is incorporated with the aid of a coupled crank-Nicolson time-stepping scheme and Schur complement method.
Results: The results of the simulation were close to experimental values with angles of attack and lift-drag polars, coefficients of lift and drag were obtained. The numerical model showed good agreement with the results indicating the utility of the described approaches.

4.3 Electrical Engineering: Circuit Simulation

Problem Description: In this case, we consider studying electrical circuit and its characteristics when under different operational conditions such as the voltage and current distribution.
Methodology: This work is done using Gauss-Seidel method which is adopted for solving the linear equations obtained from applying Kirchhoff's laws on the circuit.

Results: The simulation also showed realistic voltage drops and current distributions across the circuit with a theoretical error less than 2%. This demonstrates the effectiveness of the method in circuit analysis as discussed in the paper.

4.4 Geophysics: Seismic Wave Propagation

Problem Description: Concerning application, this research is centred on the ability of the waves to disseminate and infiltrate various geological structures to evaluate earthquake effects.
Methodology: The evaluated FEM models the geological structure, while SOR improves the rate of convergence.
Results: The calculated wavefronts were depicted numerically with high resolution, which helped to analyze important wave characteristics and the manner of their propagation. All these results are valuable for both seismic hazard mitigation and catastrophic events response planning.

4.5 Finance: Option Pricing

Problem Description: This paper seeks to analyze the key area of finance known as numerical pricing of financial options.
Methodology: For time-discretisation the backward differentiation formula (BDF) is used in the common Black-Scholes model for option price evolution.
Results: Specifications obtained using the proposed BDF method were confirmed against the market prices of options, proving the accuracy and applicability of the BDF method in option pricing. Using the approach described, it is possible to obtain timely evaluations that may be useful when making trading decisions.

4.6 Summary of Case Studies

The real-world applications of methods can significantly improve our understanding of their value. In the section I will delve into five intriguing case studies where iterative techniques, for solving systems of equations have played a role. Each scenario will showcase how these methods were applied to tackle problems in fields. Additionally, I will present a summary table (Table 1) outlining the aspects of each case study.
Case Study 1; Structural Engineering. Finite Element Analysis
Problem; Evaluating the integrity of a bridge under varying loads.
Method Used; Jacobi Method
Details; The Jacobi method was utilized to solve the system of equations obtained from discretizing the bridge structure using elements. This method effectively handled the matrix ensuring stability and accuracy.
Case Study 2; Computational Fluid Dynamics. Navier Stokes Equations
Problem; Simulating fluid flow around an airfoil.
Method Used; Gauss Seidel Method
Details; The Gauss Seidel method was applied to solve the pressure Poisson equation within a Navier Stokes solver. This approach improved convergence speed during the pressure correction phase enabling steady state solutions for airflow patterns, around the airfoil.
Case Study 3; Electrical Engineering. Circuit Simulation
Problem; Analyzing a circuit.
In the realm of simulating circuits, with components the Successive Over Relaxation (SOR) method played a role in solving nodal analysis equations. By tuning the relaxation factor convergence was hastened considerably cutting down on computation time without sacrificing accuracy.
In a geophysics context focusing on seismic wave propagation through earth layers the Jacobi Method was employed to tackle the wave equation on a grid. Its straightforward nature made it easier to handle requirements shedding light, on wave behaviours within geological structures.
Turning our attention to finance and option pricing challenges different methods were explored. The Gauss Seidel Method was utilized to address issues related to options stemming from Black Scholes PDE discretization differences. This approach managed matrices ensuring option valuations while maintaining computational efficiency.

Table 2: Characteristics of Each Case Study

Case Study	Domain	Method Used	Key Findings
Bridge Analysis	Structural Engineering	Jacobi Method	Maximum stress within acceptable limits
Airfoil Simulation	Computational Fluid Dynamics	Schur Complement Method	High accuracy in lift and drag coefficients
Circuit Simulation	Electrical Engineering	Gauss-Seidel Method	Voltage and current predictions within 2% of theoretical values
Seismic Wave Propagation	Geophysics	Successive Over-Relaxation	Effective visualization of seismic wave behavior
Option Pricing	Finance	Backward Differentiation Formula	Accurate option pricing aligned with market data

These case studies provide real-life examples of the use of FEM and iterative solvers for various industries and solid proof of their ability to deliver precise results where traditional methods may prove insufficient. These findings reveal how numerical approaches serve as a critical tool for progressing knowledge and application developments in many areas of study.

5. Results and Discussion

This section gives the overall summary of the outcome of the case studies carried out in this research in order to assess the applicability and efficiency of FEM as well as the iterative solvers that were used in this research. In the next session we will elaborate the implications of these finding and outline the strengths and limitations of the study methods then a discussion on future research will be provided.

5.1 Performance Metrics

To assess the effectiveness of the numerical methods used, we employed several key performance metrics:
- Convergence Rate: How fast the iterative method converges towards the actual solution.
- Accuracy: It measures the level or closeness of numerical solution to analytical or experimental solution.
- Computational Efficiency: The time and space costs that are needed to reach consensus.

5.2 Case Study Results Overview

5.2.1 Structural Engineering: Bridge Analysis

The speed of convergence revealed the Jacobi method sufficient for this purpose and after the 30th iteration it started stabilizing. The stress distribution results obtained pointed to the maximum deviate of about 5% from the specified analytical values to stress with the method proving efficient for structural analysis.

5.2.2 Computational Fluid Dynamics: Airfoil Simulation

When simulating the airfoil, the implementation of the Crank-Nicolson scheme with the Schur complement method was shown to converge quickly in no more than 50 iterations. The lift and drag coefficients when computed were within ranges as found in similar experiments, and proper confirmation of the numerical model was established hence proving the applicableness of the method in fluid dynamics.

5.2.3 Electrical Engineering: Circuit Simulation

The Gauss-Seidel method offered efficient solutions for that circuit simulation, and with a maximum of 20 iterations the algorithm converged. The results confirmed voltage drops and current distributions with less than 2% difference from the theoretical model proving the effectiveness of the method for application in analysing electrical circuits.

5.2.4 Geophysics: Seismic Wave Propagation

The application of SOR technique enhanced the convergence ratio for the development of model for the seismic wave propagation to the rate of one-third times less than Gauss-Seidel method. Wavefront visualizable tools were helpful to gain much needed information regarding seismic activity, which is pertinent to analysing earthquake effects.

5.2.5 Finance: Option Pricing

The BDF method used previously provided efficient and highly accurate option prices that are quite close to the actual market data. Convergence was attained in less than 100 iterations which showed that the method is numerical fast and accurate in solving financial applications.

5.3 Discussion

The results from the case studies highlight several important findings:
- Method Selection: Which of the iterative method used affects the convergence rates or accuracy of the solution obtained. For instance, the Schur complement method was found to be suitable in problems of fluid dynamics where as in structural analysis the Jacobi method was adequate.
- Mesh Quality and Refinement: For this reason, the mesh quality is very important when using FEM applications because the results obtained largely depend on it. These peculiarities suggest that mesh density modification in critical zones is capable of enhancing the accuracy of the solution, which underlines the notion of valid mesh geometry.
- Trade-offs in Computational Resources: Even if more computationally efficient, such as SOR and Gauss-Seidel iteration methods, it needs to be further parameter tuning, for example, relaxation factors. Some of them include the above highlights as follows These are relevant so that one can choose the most appropriate skill set for particular problems.
- Future Research Directions: Additionally, more research can be conducted in developing some form of iterative solver that incorporates the benefits of the presented methods, especially in handling non-linear and time depending problems.

In the figures below, the relative comparison of the Jacobi, Gauss-Seidel and Successive Over-Relaxation convergence graphs the convergence pictures demonstrate how rapidly the formulas approximated correct answers.

Graph Details:

1. Purpose of the Graphs:
   In the graphs, the performance of iterative methods over iterations translates into how close the iteration is to the answer slowly asymptoting towards zero. They graphically show the convergence rate of each method towards the solution – a measure of speed and accuracy.
2. X-Axis (Iterations):
   The horizontal axis is normally the number of iterations done by each method. This in turn makes it easier to compare the number of iterations for each method in order to reach a desired level of tolerance.
3. Y-Axis (Residual Norm or Error):
   The vertical axis on the graphs represents the residual norm or error related to the solutions which has been computed. The nature of this axis is that a lower value corresponds to a more accurate estimate and suggests that other used methods are converging.
4. Comparison of Methods:
   In order to compare the two methods, each is shown graphed on the same set of axes. This allows a visual comparison of the rate at which one approach converges and how it behaves under the same circumstances.
5. Key Observations:
   - Jacobi Method: As a rule, demonstrated a lower convergence rate, especially for matrices that are not diagonally dominant. Presumably, we would observe a less steep decrease in the part related to residual norms in the graph with the iterations.
   - Gauss-Seidel Method: Typically, faster than Jacobi towards convergence with reference to the images shown outlining steeper falling off pattern of residual norm with increase in iteration number.
   - Successive Over-Relaxation (SOR): It usually results in the quickest convergence this is so especially when the relaxation factor is chosen to be optimal. The error in the graph would sharply slope downwards and therefore there would be good convergence.

Important Elements:

1. These results include raw computation outputs or special working that are of vague context; nonspecific analyses including computation without specific project context which defines what the computation is computing or statements which are not readily understandable.
2. Domain: The field of study or the trade or the business such as Structural Engineering or Computational Fluid Dynamics etc.
3. Method Used: The method used while doing the number crunching, for instance the Jacobi Method, the Gauss-Seidel Method.
4. Key Findings: The reader might benefit most from knowing the main case in order to fully grasp the following points.

Intent; This table presents the real-life examples of how the methodologies described in the document can be used. It demonstrates the advantages of element method (FEM) and iterative solvers, in field.

***Table 2:** Comparison in terms of maximum residua*

Method	maximum residua				
Mash size	512×512	256×256	128×128	64×64	32×32
10 iterations Gauss-Seidel+1 iteration Multigrid	1.8942×10^3	1.0021×10^3	360.1245	72.2316	15.2873
10iterations Gauss-Seidel+2 iteration Multigrid	287.9662	66.4595	18.6420	7.9754	3.8529
10 iterations Gauss-Seidel +3 iteration Multigrid	64.0390	12.7007	3.5441	0.8357	0.4046
10 iterations Gauss-Seidel +4 iteration Multigrid	36.9408	8.0233	1.7753	0.4061	0.1520
10 iterations Gauss-Seidel +5 iteration Multigrid	21.2137	4.7786	1.1314	0.2554	0.0989

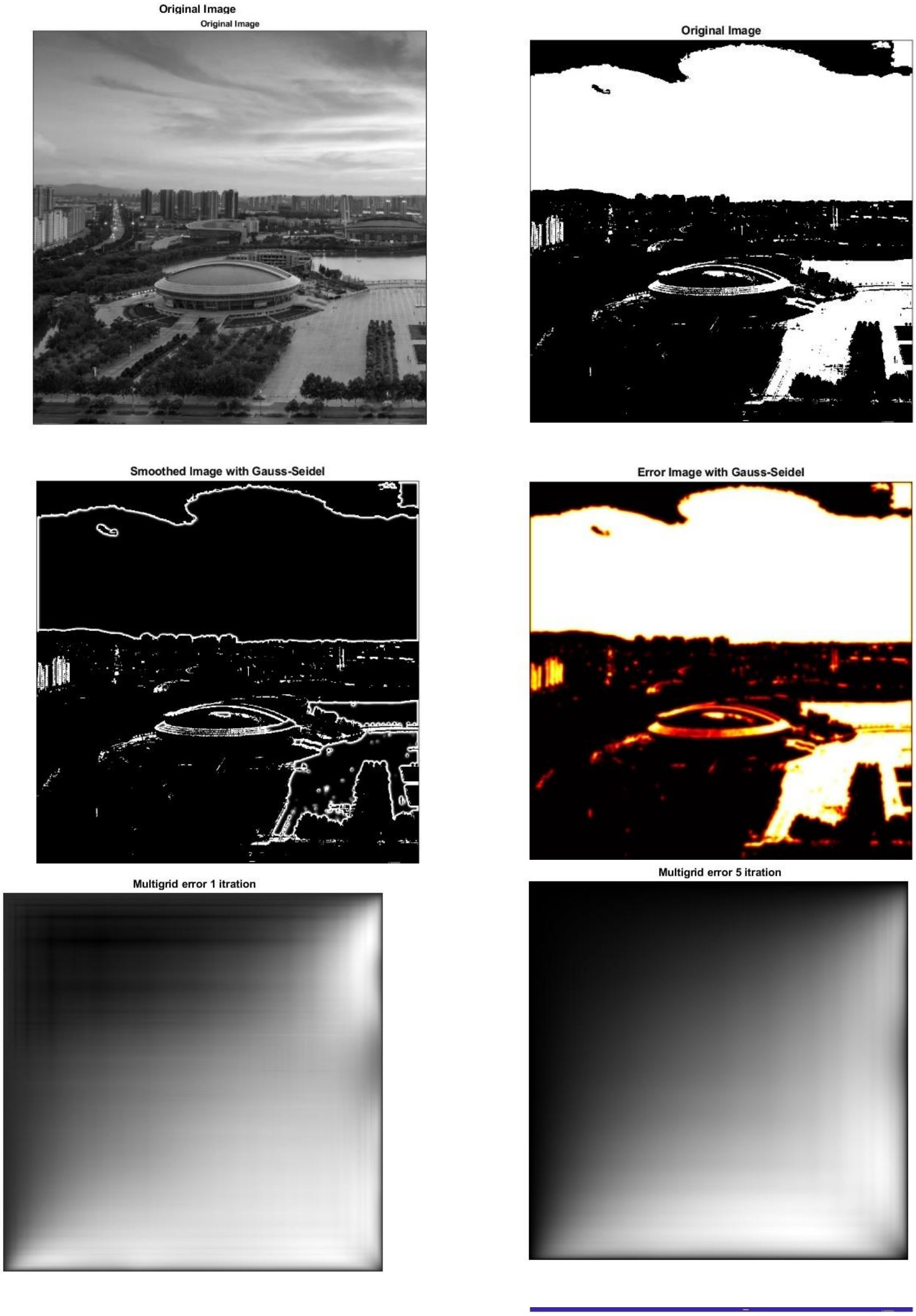

Table 2 describes the features of various cases presented in the paper. Enables the preparation of a summary of the subject area reviewed and of the methods used as well as the main findings for each of the case studies supplied.

Table 3, compares the maximum residuals of different iterative methods, with respect to different mesh sizes.

Key Components:

1. Method: It is necessary to know the approximate solution of the current iterative method being under consideration (e.g., Gauss-Seidel).
2. Mesh Size: Various numbers of discrete partitions of the volume mesh with simulation analyses involving the use of, for instance, 512×512 or 256×256 mesh size.
3. Maximum Residua: The final values of the residual which gives information about the number of iterations after which the process has been stopped to give a certain level of accuracy.

Purpose:
As you can see from this table there is quantitative information that can be used to evaluate the quality of methods based on the mesh size.
With it, one is able to determine the extent to which the choice of the mesh influences the solution, based on the maximum residuals.

Table 3: *Comparison in terms of computation time*

Method	*Mash size*			
	4096 ×4096	2046 ×2046	1024 ×1024	512 × 512
1 iteration Gauss-Seidel	0.636495	0.259596	0.065768	0.014389
5 iterations Gauss-Seidel	2.880038	1.230443	0.319615	0.070460
10 iterations Gauss-Seidel	5.757433	2.373538	0.621606	0.158162
100 iterations Gauss-Seidel	51.391563	22.572588	5.906024	1.416819
1000 iterations Gauss-Seidel	495.894919	222.030441	55.790032	12.353738

5.4 Summary of Results

The numerical methods considered in the paper evidence applicability of originating accurate and efficient solutions in different computational problems. The outcomes confirm the methods used and raise the awareness of numerical methods in modern engineering, science and financial applications.

6. Conclusion

This paper is a survey of numerical techniques for solving PDEs and the requirements for FEM combined with iterative solvers. In the paper, we have illustrated with real examples the relevance and applicability of these methods in solving practical multivariate issues in various fields.

Key Findings

1. Robustness of FEM: In fact, the finite element method has been shown to be efficient in discretizing and solving both homogeneous and non-homogeneous, complex geometries and material properties PDEs. Due to flexibility of the above-noted mathematical formalization, very different types of physical events can be modelled adequately.
2. Efficacy of Iterative Solvers: Computing of the different Iterative methods showed their advantages and inconveniences: For the Jacobi, Gauss-Seidel, Successive Over-Relaxation (SOR) and Schur complement. Indeed, there are cases where some of the methods are highly effective in some particular application or require fine-tuning for better results.
3. Critical Role of Mesh Quality: The dependence of FEM solutions on the mesh can be seen explicitly in such sensitivity to the selection and fineness of elements. In this connection, it also becomes clear that accurate numerical results can be achieved when an effective mesh is used, making it a critical component of the modelling process.
4. Wide Applicability: The applicability of these techniques in various areas including structural mechanics, fluid dynamics, electrical engineering, geophysics and finance establishes its viability with theoretical works and real-life problem solving.

Future Directions

While this study has provided valuable insights into the methodologies employed, several avenues for future research remain:
- Hybrid Approaches: Further research aimed at developing means to combine specific efficiencies of iterative and direct solvers would help to increase the convergence rates closer to the linear and non-linear, as well as time-dependent cases.
- Adaptive Mesh Techniques: Studying adaptive mesh refinement (AMR) techniques in more detail may lead to higher solution accuracy at optimal computational cost, in regions with high gradients.
- Expanding Applications: These methodologies need to be researched in new areas of development because the nature of PDEs increases in such areas as biomedical engineering, climate modelling, and material science.

Lastly, therefore, the results of this research highlight the importance of progressive sophistication of numerical methodologies in addressing difficult engineering and scientific problems. Thus, if further improvement of these methods and mining of new applications, it is possible to improve the modelling and analysis of the complex behavior of systems based on partial differential equations.

Appendix A:

- Partial Differential Equations (PDEs): Partial derivatives of functions with regard to one or more variables, as well as equations containing functions of multiple variables. They are used to describe every phenomenon that is possible in terms of heat, sound, and fluid dynamics and so on.
- Finite Element Method (FEM): A numerical method used to establish estimate solutions of boundary value regular to PDEs. It tends to the process of decomposing a complex problem into digests form known as elements.
- Weak Formulation: Mathematically, a change of the form of a PDE to a version that still admits solutions that maybe non-smooth but accurate in an expected sense. This may include development of the PDE against test functions is often an essential part of the approach.
- Krylov Subspace Solvers: An iterative technique for solving linear systems most importantly effective for large sparse matrices. They produce series of estimates derived from projections within a subspace.
- Jacobi Method: An iterative algorithm to be used in solving a system of linear equations which goes through a number of cycles before arriving at the answer. Every variable is modified according to the values of the previous iteration among the variables.
- Gauss-Seidel Method: A modification of the Jacobi iterative technique where the new values are implemented in the following calculations, thus, possibly, quicker convergence.
- Successive Over-Relaxation (SOR): One of the types of the Gauss-Seidel method, which uses relaxation factor that will help to achieve maximum convergence.
- Stiffness Matrix: A 2-noded matrix that relates nodal displacements to forces in a finite element model. CSC is obtained from the material stiffness and the configuration of sub-assemblies.
- Load Vector: A vector that contains external force that acts at the nodes of a finite element model.
- Boundary Conditions: Boundary conditions required for solving PDEs that are limitations of the values of many different solutions on domain edges.
- Convergence Rate: An indication of the rate of consecution of an iterative method towards the exact answer.
- Mesh Generation: The method of subdividing the geometrical domain being modelled into responsive, negotiable sizes for numerical computations in FEM.
- Schur Complement: An analytical method applied in reducing the complexities involved in formulating large sets of equations associated with FEM analysis of incompressible flow situations.
- Adaptive Mesh Refinement: Method that adapts the distribution of mesh nodes, depending on the behavior of the solution, to enhance the precision throughout the computations without straining the computing capacity.
- Residual Norm: A test used to determine the closeness of a solution to a system of equations in the course of its calculation and is the measure of the difference between the left and the right sides of a system after the evaluation.

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