Characterizations of Pairwise Contra Pre-continuous

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Submission data: 5/6/2024 Electronic publishing data: 7/9/2024

Abstract. The aim of this paper is to study the notion of contra pre-continuous functions in a bitopological space called pairwise contra pre- continuous up on some generalizations of closed sets, namely pairwise pre-closed and pairwise pre-open sets. Also, we introduce pairwise contra pre-closed graphs and study some of their properties. In addition, we study the implications of these types of functions on some covering properties.

Keywords: Bitopological space; pairwise pre-open sets; pairwise contra pre-continuity; pairwise contra pre-closed graphs; pairwise submaximallty space; pairwise strongly covering properties.

1- Introduction and Preliminaries

The study of some generalizations about closed sets in topological space known as pre-closed sets was started by several authors in recent years [4,12]. Mashhour et. al. proposed the idea of a pre-open set in topological spaces [12]. The subset A of a space X is called pre-open if $A \subset int(cl(A))$. So, A is termed pre-open set in a space X. In addition, the pre-closed set is defined as the complement of a pre-open set. Based on the sets recall mentioned above, many researchers gave some kinds of functions in topological space. For example, Dontchev presented the concept of contra continuity [4]. Also, Jafri and Noiri introduced and investigated the concept of contra pre-continuity in topology which is weaker than contra-continuity [5]. A function $f: (X, \sigma) \to (Y, \delta)$ is termed contra pre-continuous if V is δ-open set then $f^{-1}(V)$ is σ-pre-closed in X. They obtained many basic properties and some effects of it on some topological concepts as connectedness and covering properties as compactness. Jelić introduced the idea of pre-open sets in bitopology [6]. Also, some researchers studied various types of functions and graphs in bitopological spaces [2,6,8] and others).

In the current work, we focus on the notion of contra pre-continuous in bitoplogical settings and provide several characterizations of this function. Furthermore, we investigate its graphs and the effects on many covering properties as pairwise strongly Lindelöf spaces.

During this study, all spaces (X, τ) and (X, σ_1, σ_2) all the time mean topological spaces and bitopological spaces, respectively. In current investigation, we practice the symbols (σ_1, σ_2)- to signify assured properties with respect to topology σ_i and σ_j as bitopological spaces, such that $i, j = \{1, 2\}$. Using σ_i-open set, we shall imply the open set with respect to topology σ_i in X. Also, σ_i-open cover of X implies that the cover of X by σ_i-open sets in X. The reader may find more details about symbols and thoughts in [7].

Definition 1.1. [6] A set P in (X, σ_1, σ_2) is termed (i, j)-pre-open if and only if $P \subset i\text{–}int(j\text{–}cl(P))$. A subset P is said to be pairwise pre-open in X if it is (1,2)-pre-open and (2,1)-pre-open.

A set is named (i, j)-pre-closed if it is the complement of (i, j)-pre-open. A set P is called pairwise pre-closed in X if it is (1,2)-pre-closed and (2,1)-pre-closed.
The collection of all (i, j)-pre-open (resp. (i, j)-pre-closed) sets in X will be presented by (i, j)-PO(X) (resp. (i, j)-PC(X)).
Definition 1.2. [9] A $f: (X, \sigma_1, \sigma_2) \to (Y, \delta_1, \delta_2)$ is known as continuity if $f^{-1}(V)$ is σ_i-open in X for all δ_i-open set V in Y such that $i = 1,2$.

Definition 1.4. [3] A space (X, σ_1, σ_2) is termed (i, j)-submaximallty if any j-dense set in X is i-open. X is pairwise submaximallty if it is (1,2)- submaximallty and (2,1)- submaximallty.

Definition 1.5. [11] $f: (X, \sigma_1, \sigma_2) \to (Y, \delta_1, \delta_2)$ is (i, j)-contra continuity if $f^{-1}(V)$ is σ_i-closed in X for each δ_j-open set V in Y such that $i, j = 1,2$ and $i \neq j$. A function f is called pairwise contra continuity if it is both (1,2)-contra continuity and (2,1)-contra continuity.

2-Pairwise contra pre-continuity.

Due to Jafri definition [5], we shall generalize it to pairwise contra pre-continuity in bitopology as the following:

Definition 2.1. A function $f: (X, \sigma_1, \sigma_2) \to (Y, \delta_1, \delta_2)$ is (i, j)-contra pre-continuity if $f^{-1}(V)$ is (σ_i, σ_j)-pre-closed in X for each δ_j-open set V in Y, for $i, j = 1,2$ and $i \neq j$. The function f is said to be pairwise contra pre-continuity if it is both (1,2)-contra pre-continuity and (2,1)-contra pre-continuity.

Remark 2.1. It is clear that each pairwise contra continuity is pairwise contra pre-continuity but the converse is not true in general as the next example shows.

Example 2.1. Let $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ be a function and $X=Y=\{a_1,a_2,a_3\}$ such that $\sigma_1=\{\emptyset,X,\{a_1\},\{a_1,a_3\}\}$, $\sigma_2=\{\emptyset,X,\{a_3\}\}$, $\delta_1=\{\emptyset,Y,\{a_1\}\}$ and $\delta_2=\{\emptyset,Y,\{a_2\}\}$. Let f defined as $f(a_1)=a_1,\ f(a_2)=a_3,\ f(a_3)=a_2$. f is not pairwise contra continuity **[11]**, but it is pairwise contra pre-continuity.

Further, the notions of continuity and pairwise contra pre-continuity are independent.

Example 2.2. Let $f:(X,\sigma_1,\sigma_2)\to(X,\delta_1,\delta_2)$ be a function and $X=\{a_1,a_2,a_3\}$ such that $\sigma_1=\{\emptyset,X,\{a_2\},\{a_1,a_2\}\}$, $\sigma_2=\{\emptyset,X,\{a_2,a_3\}\}$, $\delta_1=\{\emptyset,X,\{a_1\}\}$, $\delta_2=\{\emptyset,X,\{a_2\}\}$. Let f defined as $f(a_1)=a_1,\ f(a_2)=a_3,\ f(a_3)=a_2$. Thus f is pairwise contra continuity **[11]**. So, it is pairwise contra pre-continuity. f is not continuity since $V=\{a_1\}\in\delta_1$ but $f^{-1}(\{a_1\})=\{a_1\}\notin\sigma_1$.

Theorem 2.1. Let $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$. The followings are equivalent:
(1) f is (i,j)-contra pre-continuity;
(2) $f^{-1}(W)$ is (σ_i,σ_j)-pre-open set in X, for any δ_j-closed set W in Y;
(3) for every point x in X and every δ_j-closed set W containing $f(x)$, we have (σ_i,σ_j)-pre-open set V containing x such that $f(V)\subset W$.

Proof: (1) $\Longrightarrow$ (2) Suppose that W be any δ_j-closed set in Y. So, $Y\backslash W$ is δ_j-open. Since f is (i_i,j)-contra pre-continuous, $f^{-1}(Y\backslash W)$ is (σ_i,σ_j)-pre-closed. Then, $f^{-1}(Y\backslash W)=X\backslash f^{-1}(W)$ is i-(σ_i,σ_j)-pre-closed and $f^{-1}(W)$ is (σ_i,σ_j)-pre-open set in X.

(2) $\Longrightarrow$ (1) Suppose that V be any δ_i-open set. So, $Y\backslash V$ is δ_j-closed set. By (2), we have $f^{-1}(Y\backslash V)=X\backslash f^{-1}(V)$ is (σ_i,σ_j)-pre-open set. This implies that $f^{-1}(V)$ is (σ_i,σ_j)-pre-closed set. Therefore, f is (i,j)-contra pre-continuity.

(2) $\Longrightarrow$ (3) Assume that W be δ_j-closed set containing $f(x)$. Via (2), $f^{-1}(W)$ is (σ_i,σ_j)-pre-open containing x such that $f^{-1}(W)=U\Rightarrow f(U)\subseteq W$.

(3) $\Longrightarrow$ (2) Suppose that W be any δ_j-closed set such that $f(x)\in W \implies x\in f^{-1}(W)$. Thus, we have $V_x\in(\sigma_i,\sigma_j)$-$PO(X,x)$ such that $f(V_x)\subset W$. Then, $f^{-1}(W)=\cup\{V_x: x\in f^{-1}(W)\}\in(\sigma_i,\sigma_j)$-$PO(X,x)$.■

Here, we introduce the notion of pairwise pre-connected as following:

Definition 2.2. A space (X,σ_1,σ_2) is said to be (i,j)-pre-connected if X cannot be written as union of (i,j)-pre-open set and (j,i)-pre-open set in X. X is called pairwise pre-connected if it is (1,2)-pre-connected and (2,1)- pre-connected.

Theorem 2.2. If $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ is pairwise contra pre-continuity onto and X is (σ_i,σ_j)-pre-connected, then Y is (δ_i,δ_j)-connected.

Proof: Suppose that Y is not (δ_i,δ_j)-connected. So, we have δ_i-open set W_1 and δ_j-open set W_2 such that $W_1\cup W_2=Y$ and $W_1\cap W_2=\emptyset$ [7]. Furthermore, W_1 is δ_i-closed set and W_2 is δ_j- closed set. Due to pairwise contra pre-continuity of f, $f^{-1}(W_1)$ is (j,i)-pre-open and $f^{-1}(W_2)$ is (i,j)-pre-open in X. As f is onto, we get

$$f^{-1}(W_1)\cup f^{-1}(W_2)=X$$

and

$$f^{-1}(W_1)\cap f^{-1}(W_2)=\emptyset.$$

This contracts that X is (σ_i,σ_j)-pre-connected space. Therefore Y is (δ_i,δ_j)-connected space.■

Theorem 2.3. [3] Every subspace A of (i,j)-submaximallty space X is (i_A,j_A)-submaximallty.

Theorem 2.4. If a bitopological space X is (i,j)-submaximallty, then each (i,j)-pre-open subset of X is i-open.

Proof: From **Theorem 2.3**. ■

Theorem 2.5. If $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ is (i,j)-contra pre-continuity and X is (i,j)- submaximllty, then f is (i,j)-contra continuity.

Proof: Assume that V be δ_j-open set. So $f^{-1}(V)$ is a (σ_i,σ_j)-pre-closed set in X because f is (i_i,j)–contra pre-continuity. So $X\backslash f^{-1}(V)$ is (σ_i,σ_j)-pre-open set. As X is (σ_i,σ_j)- submaximllty space and by **Theorem 2.4**, $X\backslash f^{-1}(V)$ is σ_i-open. Then $f^{-1}(V)$ is σ_i-closed set in X. This shows that f is (i_i,j)-contra continuity.■

Theorem 2.6. Let $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ be (i,j)-contra pre-continuity and $g:(Y,\delta_1,\delta_2)\to(Z,\gamma_1,\gamma_2)$ is j-continuity, thus $gof:(X,\sigma_1,\sigma_2)\to(Z,\gamma_1,\gamma_2)$ is (i_i,j)-contra pre-continuity.

Proof: Assume that W be γ_j-open set. Since g is j-continuity, then $g^{-1}(W)$ is δ_j-open set. As f is (i,j)-contra pre-continuity, then $f^{-1}(g^{-1}(W))$ is (σ_i,σ_j)-pre-closed set in X. Due to $f^{-1}(g^{-1}(W)=(gof)^{-1}(W)$, so $gof:(X,\sigma_1,\sigma_2)\to(Z,\gamma_1,\gamma_2)$ is (i,j)-contra pre-continuity. ■

Next, we state the restriction of pairwise contra pre-continuity.

Theorem 2.7. Let $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ be (i_i,j)-contra pre-continuity. If U is (σ_1,σ_2)-pre-closed subset of X then the restriction $f_U:(U,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ is (i_i,j)-contra pre-continuity.

Proof: Suppose that V is δ_j-open in Y. $f^{-1}(V)$ is (σ_i,σ_j)-pre-closed in X since f be (i_i,j)-contra pre-continuity. Thus $f^{-1}(V)\cap U$ is (σ_i,σ_j)-pre-closed in U. Since $(f_U)^{-1}(V)=f^{-1}(V)\cap U$ is (σ_i,σ_j)-pre-closed in U. Hence f_U is (i_i,j)-contra pre-continuity.■

Definition 2.3. The graph $G(f)$ of a function $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ is called (i,j)-contra pre-closed if for every point $(x,y)\in(X\times Y)\backslash G(f)$, there exist $M\in(\sigma_i,\sigma_j)$- $PO(X,x)$ and N is δ_j-closed set in Y such that $(M\times N)\cap G(f)=\emptyset$.

Lemma 2.1. The graph $G(f)$ of the function $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ is (i_i,j)-contra pre-closed if and only if for any point $(x,y)\in(X\times Y)\backslash G(f)$, there exist $M\in(\sigma_i,\sigma_j)$-$PO(X,x)$ and $N\in\delta_j$-$C(Y,y)$ such that $f(M)\cap N=\emptyset$.

Proof: $\Longrightarrow$ Since $(x,y)\notin G(f)$, there are two subsets $M\in(\sigma_i,\sigma_j)$-$PO(X,x)$ and $N\in\delta_j$-$C(Y,y)$ containing x and y respectively such that $(M\times N)\cap G(f)=\emptyset\Longrightarrow f(X)\cap N=\emptyset$. So we get $f(M)\cap N=\emptyset$.

$\Leftarrow$ We will prove that the graph is (i_i,j)-contra pre-closed. For $(x,y)\in(X\times Y)\backslash G(f)$, we have $y\neq f(x)$ for every $x\in X$. By assumption, there exist two sets $M\in(\sigma_i,\sigma_j)$-$PO(X,x)$ and $N\in\delta_j$-$C(Y,y)$ such that $y\in N\Longrightarrow y\in Y\backslash f(X)\Longrightarrow N\cap f(X)=\emptyset$. For (σ_i,σ_j)-pre-open set M in X, we have $(M\times N)\cap(X\times f(X))=\emptyset\Longrightarrow(M\times N)\cap G(f)=\emptyset$. Therefore, $G(f)$ is (i_i,j)-contra pre-closed in $X\times Y$. ■

Theorem 2.8. Let $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ be (i,j)-contra pre-continuity and Y is (δ_i,δ_j)-Urysohn space, then $G(f)$ is (i,j)-contra pre-closed in $X\times Y$.

Proof: Let $(x,y)\in(X\times Y)\setminus G(f)$. Due to $y\neq f(x)$ and Y is (δ_i,δ_j)-Urysohn, we have δ_i-open sets $V$and W such that $f(x)\in V,y\in W$and δ_j-$cl(V)\cap\delta_j$-$cl(W)=\emptyset$. Asf is (i,j)-contra pre-continuity, there exists (σ_i,σ_j)-pre-open set containing x such that $f(U)\subset\delta_j$- $cl(V)$. Then, we get $f(U)\cap\delta_j$- $cl(W)=\emptyset$. This shows that $G(f)$ is (i,j)-contra pre-closed. ■

3-Effects of pairwise contra-pre-continuous on some pairwise strongly covering properties.

Several kinds of lindelöfness were generalized and studied in a bitopological space defined by pairwise pre-open and i –open sets [1,10].

Now, we shall define (i,j)-strong Lindelöf spaces in sense of Kiliçman definition of p_1-Lindelöf spaces [10].

Definition 3.1. Let (X , σ_1,σ_2) be a bitopological space, then X is called:

(1) (i,j)-strongly Lindelöf if any (i,j)-pre- open cover of X has j-open countable subcover. X is called p_1- strongly Lindelöf space if it is (1,2)-strongly Lindelöf and (2,1)-strongly Lindelöf.

(2) strongly i-L-closed if any i-closed cover of X has i-closed countable subcover. X is called strongly L-closed space if it is strongly i-L-closed for $i=1,2$.

Remark 3.1 Every p_1--strongly Lindelöf space is p_1-Lindelöf.

Example 3.1 Let X be real numbers and σ_1=σ_2= discrete topology. (X,σ_1,σ_2) is p_1-strongly Lindelöf space. Therefore, X is p_1-Lindelöf space.

Theorem 3.1. If (X , σ_1,σ_2) is σ_i-Lindelöf whit respect to σ_j and (i,j)-submaxmallty space, then it is (i,j)-strongly Lindelöf space.

Proof: It is immediately from the definition of σ_i Lindelöf whit respect to σ_j [10] and **Theorem 2.4.** ■

Definition 3.2. A subset A of a space (X , σ_1,σ_2) is said to be:

(1) (i,j)-strongly Lindelöf relative to X if each cover of A by (i,j)-pre-open sets of X has j-open countable subcover. A subset A is called p_1-strongly Lindelöf relative to X if it is (1,2)-strongly Lindelöf relative to X and (2,1)-strongly Lindelöf relative to X.

(2) strongly$i$-L-closed relative to X if the subspace A is strongly$i$-L-closed. A subset A is called strongly$L$-closed relative to X if it is strongly i-L-closed relative to X for $i=1,2$.

Definition 3.3. (X,σ_1,σ_2) is called (i,j)-P_p-space if the intersection of countable of (i,j) –pre-open sets is (j,i)–pre-open set.

Theorem 3.2. Let (X,σ_1,σ_2) be (σ_j,σ_i)-submaxmallty and (σ_i,σ_j)-P_p-space. If $f:(X,\sigma_1,\sigma_2)\to(Y,\delta_1,\delta_2)$ has (i,j)-contra pre-closed graph, so, for any strongly δ_j-L-closed set R, $f^{-1}(R)$ is σ_j-closed set .

Proof: Suppose that R be strongly δ_j-L-closed set in Y and $a\notin f^{-1}(R)$. For any $r\in R$, $(a,r)\notin G(f)$. Via **Lemma 2.1**, we have (σ_i,σ_j)-pre-open set U_r containing a and δ_j-closed set V_r containing r such that $f(U_r)\cap V_r=\emptyset$ whereas $\{R\cap V_r: r\in R\}$ is δ_j-

closed cover of the subspace R, there exists a δ_j-closed countable subcover $\{V_{r_n}: n \in \mathbb{N}\}$ such that

$$R \subset \bigcup_{n\in\mathbb{N}} V_{r_n}.$$

As X is (σ_i, σ_j)-P_p-space, so the set $M = \bigcap\{U_{r_n}: n \in \mathbb{N}\}$ is (σ_j, σ_i)-pre-open set. Because X is (σ_j, σ_i)-submaximallty, M is σ_j-open set by **Theorem 2.4**. Therefore $f(M) \cap R = \emptyset \Longrightarrow M \cap f^{-1}(R) = \emptyset$. Thus, $f^{-1}(R)$ is σ_j-closed in X. ■

Theorem 3.3. If $f: (X, \sigma_1, \sigma_2) \to (Y, \delta_1, \delta_2)$ is (i, j)-contra pre-continuity and R is (σ_i, σ_j)-strongly Lindelöf relative to X, then $f(R)$ is strongly δ_j-L-closed in Y.

Proof. Assume that the family $\{O_\alpha: \alpha \in I\}$ be any cover of $f(R)$ via δ_j-closed sets of the subspace $f(R)$. For any $\alpha \in I$, there is δ_j-closed setQ_α in Y such that $O_\alpha = Q_\alpha \cap f(R)$. For every point $r \in R$, there exists $\alpha_r \in \nabla$ such that $f(r) \in Q_{\alpha_r}$ .By **Theorem 2.1,** there exists $M_r \in (\sigma_i, \sigma_j)$-$PO(R, r)$ such that $f\ (M_r) \subset Q_{\alpha_r}$. Since the family $\{M_r: r \in R\}$ is (σ_i, σ_j)-pre-open cover of R, there exists σ_j-open countable subcover $\{M_{r_n}: n \in \mathbb{N}\}$ such that $R \subset \bigcup_{n\in\mathbb{N}} M_{r_n}$. Therefore, we get

$$f(R) \subset \bigcup\{f(M_{r_n}): n \in \mathbb{N}\}$$
$$\subset \bigcup\{Q_{\alpha_{r_n}}: n \in \mathbb{N}\}.$$

This implies that $f(R)$ is strongly δ_j-L-closed in Y. ■

4-conclusion.

During this work, we studied the concept of contra pre-continuity in bitopological spaces. Also, we obtained pairwise contra pre-closed graphs and their properties. In addition, we investigated the implications of these types of functions on some covering properties.

References

[1] Almuhur E., AL-labadi M. (2021). Pairwise strong Lindelöf, pairwise nearly, almost and weakly Lindelöf bitopological spaces. Wseas Transactions on Mathematics. Vol.(20),152-158.

[2] Bouseliana Hend M., Kiliçman A. (2022). Implications of some types of pairwise closed graphs. Proyecciones Journal of Mathematics. Vol. 41, N 5, pp. 1131-1139.

[3] Dochviri I. (2010).On submaximality of bitopological spaces, Kochi J. Math. 5, 121–128.

[4] Dontchev J, (1996). Contra-continuous functions and strongly S-closed spaces. Int. Math Sci, 19 303-10.

[5] Jafari S., Noiri T. (2002). On Contra-Precontinuous Functions, Bull. Malaysian Math. Sc. Soc. (Second Series) **25** , 115-128.

[6] Jelic M. (1990). A Decomposition of Pairwise Continuity. J. Inst. Math. Comput. Sci. Math. Ser., 3, 25-29.

[7] Kelly J. C. (1963). Bitopological spaces, Proc. London Math. Soc. 1**3** (3): 71 - 89.

[8] Khedr, F. H., Al-Areefi , S. M., Noiri T. (1992). Precontinuity and Semi-Precontinuity in Bitopological Spaces, Indian J. pure appl. Math., 23(9):625-633.

[9] Kiliçman A., Salleh Z. (2007). Mappings and Pairwise Continuity on Pairwise Lindelöf Bitopological Spaces, Albanian J. Math., **1**(2), 115–120.

[10] Kiliçman A., Salleh Z. (2007). On pairwise Lindelöf bitopological spaces, Topology and its Applications (**154**), 1600–1607.

[11] Mahmood P., Mustafa, K., A., Mohammed J. (2013). Contra-Continuous Functions in Bitopological Spaces, Tikrit Journal of Pure Science 18 (3), 172-177.

[12] Mashhour A. S., El-Monsef M. E. , El-Deeb S. N. (1982). On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, **53**, 47–53.

[13] Öztürk U., Hussein R., Kharatal A. (2022). New results for certain types of contra pre-continuous functions in topological space. Journal of Physics: Conference Series.