SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY GENERALISED DERIVATIVE OPERATOR

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Submission data: 29/04/2024 Electronic publishing data: 29/7/2024

Abstract: In this paper, we obtain the Fekete-szego inequality defined on the class of analytical functions normalized in the unit disk, which is defined by generalized derivative operator and the Hadamard product with a normalized analytic function. The main purpose of this paper is to give an estimate for the Fekete szego inequality when $f \in \Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$.

Keywords: Analytical functions**,** Derivative operator, Fekete–Szego inequality, Hadamard product.

Introduction:

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which is analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \ ; \ |z| < 1\},$$

and satisfy the normalization condition

$$f(0) = 0 \quad , \quad \acute{f}(0) = 1.$$

and

$g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, $\varphi(z) = z + \sum_{k=2}^{\infty} w_k z^k$, and $\psi(z) = z + \sum_{k=2}^{\infty} d_k z^k$ be analytic the functions in $\mathbb{D}$ where $b_k, w_k, d_k > 0$ and $w_k > d_k$, and we defined the Hadamard product as follows :

$$g(z) * \varphi(z) = z + \sum_{k=2}^{\infty} b_k w_k z^k,$$
$$g(z) * \psi(z) = z + \sum_{k=2}^{\infty} b_k d_k z^k. \quad (2)$$

Now we define the class by $f \in \Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$, as follows.

Definition 1[1]:

Let the function f be given by (1). Then, the function $f \in \Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$; $n,m \in \mathbb{N}_0 = \{0,1,2,\dots.\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0, 0 \leq \alpha < 1, 0 \leq \beta < 1,$ and $0 \leq \gamma < 1$ if and only if there exists $g \in \mathcal{A}$, $g(z) \neq 0$ such that:

$$Re\left(\frac{\alpha z^2\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{g(z)} + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{g(z)}\right) > \beta, \quad (3)$$

$$Re\left(\frac{g(z)*\varphi(z)}{g(z)*\psi(z)}\right) > \gamma, \quad \text{for } z \in \mathbb{D} \quad (4)$$

for some $\varphi(z)$ and $\psi(z)$ both is analytic in $\mathbb{D}$ such that $g(z) * \psi(z) \neq 0$, $w_k, d_k > 0$ and $w_k > d_k$, $k \geq 2$.

In order to derive the generalized derivative operator, we define the analytic function

$$\phi^m(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \quad (5)$$

where $m \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and $\lambda_2 , \lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$.

Now, we introduce the new generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ as the following:

Definition 2 [2]:

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n): A \longrightarrow A$

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = \phi^m(\lambda_1,\lambda_2,l)(z) * R^n f(z) \quad , (z \in \mathbb{D}) \quad (6)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,\dots.\}$ and $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [3] ,and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k , (n \in \mathbb{N}_0, z \in \mathbb{D}),$$

If f is given by (1), then we easily find from the equality (6) that

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m} c(n,k) a_k z^k,$$

where $n,m \in \mathbb{N}_0 = \{0,1,2,\dots.\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Preliminary Results

Lemma 1[4]:

Let h be analytic in $\mathbb{D}$ with $Re\, h(z) > 0$ and be given by $h(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, for $z \in \mathbb{D}$, then ,

where $n \geq 1$, $\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}$. (7) $|c_n| \leq 2$

Lemma 2[5]:

Let $g \in S^*$,the starlike function with $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ .Then for μ real,

$$|b_3-\mu b_2^2| \le max\{1, |3-4\mu|\}. \quad (8)$$

The first result for the class is as follows.

Theorem 1 :
Let the function f given by (1) belong to the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$ and $0 \le \alpha < 1$. Then,

$$(\alpha+1)\mathrm{A}|a_2| \le \frac{(w_2-d_2)(1-\beta)+1-\gamma}{w_2-d_2}, \quad (9)$$

$$3(2\alpha+1)\mathrm{B}|a_3| \le \frac{4(1-\gamma)^2}{(w_3-d_3)(w_2-d_2)} + \frac{4(1-\gamma)(1-\beta)}{w_2-d_2} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta).$$

Where

$$\mathrm{A} = \frac{(1+\lambda_1+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2)^m} C(n,2),$$

$$\mathrm{B} = \frac{(1+\lambda_1(2)+l)^{m-1}}{(1+l)^{m-1}\big(1+\lambda_2(2)\big)^m} C(n,3)$$

Proof

From definition, we have

$$g(z)*\varphi(z) = (p(z)(1-\gamma)+\gamma)\big(g(z)*\psi(z)\big), \quad (10)$$

For any $z \in \mathbb{D}$, with $Rep(z) > 0$ given by $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$, where $p_1, p_2, p_3, \cdots \in \mathbb{C}$.

From (10), we have

$$z + b_2w_2z^2 + b_3w_3z^3 + \cdots = (z + b_2d_2z^2 + b_3d_3z^3 + \cdots) + (p_1(1-\gamma)z^2 + p_1(1-\gamma)b_2d_2z^3 + \cdots) + (p_2(1-\gamma)z^3 + p_2(1-\gamma)b_2d_2z^4 + \cdots). \quad (11)$$

Now, equating coefficients, we get

$$b_2(w_2-d_2) = p_1(1-\gamma), \quad (12)$$

$$b_3(w_3-d_3) = b_2d_2p_1(1-\gamma) + p_2(1-\gamma). \quad (13)$$

And also follows from (3) that

$$\alpha z^2\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)'' + z\big(I^m(\lambda_1,\lambda_2,l,n)f(z)\big)' = g(z)(h(z)(1-\beta)+\beta), \quad (14)$$

where $Reh(z) > 0$, and writing $h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$, where $c_1, c_2, c_3, \cdots \in \mathbb{C}$, and now

$$2\alpha\mathrm{A}a_2z^2 + 6\alpha\mathrm{B}a_3z^3 + \cdots + z + 2\mathrm{A}a_2z^2 + 3\mathrm{B}a_3z^3 + \cdots = (z + c_1(1-\beta)z^2 + c_2(1-\beta)z^3 + \cdots) + (b_2z^2 + c_1(1-\beta)b_2z^3 + \cdots) + (b_3z^3 + c_1(1-\beta)b_3z^4 + \cdots) + \cdots \quad (15)$$

and equating coefficients give

$$2(\alpha+1)\mathrm{A}a_2z^2 = c_1(1-\beta)z^2 + b_2z^2, \quad (16)$$

$$3(2\alpha+1)\mathrm{B}a_3z^3 = (1-\beta)(c_2+b_2c_1)z^3 + b_3z^3, \quad (17)$$

from equation (16), (17) we obtain

$$2(\alpha+1)\mathrm{A}a_2 = c_1(1-\beta) + b_2, \quad (18)$$

$$3(2\alpha+1)\mathrm{B}a_3 = (1-\beta)(c_2+b_2c_1) + b_3. \quad (19)$$

from equation (18) we find a_2^2 ,

$$\mathrm{A}a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)} + \frac{b_2}{2(\alpha+1)}$$

We get:

$$a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}} \Rightarrow a_2^2 = \left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}} + \frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2$$

The result follows applying in equalities: $|p_1| \le 2$, $|p_2| \le 2$, $|c_1| \le 2$, $|c_2| \le 2$, and from (12), (13) we get $b_2 = \frac{2(1-\gamma)}{(w_2-d_2)} \Rightarrow |b_2| \le \frac{2(1-\gamma)}{(w_2-d_2)}$, and

$$b_3 = \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}$$

$$\Rightarrow |b_3| \le \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}.$$

From (18) we obtain

$$(\alpha+1)\mathrm{A}a_2 = \frac{c_1(1-\beta)}{2} + \frac{b_2}{2} \le \frac{2(1-\beta)}{2} + \frac{2(1-\gamma)}{2(w_2-d_2)},$$

$$\therefore (\alpha+1)\mathrm{A}|a_2| \le \frac{(w_2-d_2)(1-\beta)+1-\gamma}{w_2-d_2}.$$

From (19) we obtain

$$3(2\alpha+1)\mathrm{B}|a_3| \le 2(1-\beta) + 2(1-\beta)\left(\frac{2(1-\gamma)}{2(w_2-d_2)}\right) + \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3},$$

$$3(2\alpha+1)\mathrm{B}|a_3| \le 2(1-\beta) + \frac{4(1-\beta)(1-\gamma)}{w_2-d_2} + \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} + \frac{2(1-\gamma)}{w_3-d_3}.$$

Now we display the main result for the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$.

Main Result

Theorem 2.

Let the function f be given by (1) and belong to the class $\Delta_{n,m}^{\lambda_1,\lambda_2,l}(\varphi(z),\psi(z);\alpha,\beta,\gamma)$. Then

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \begin{cases} \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} + \frac{4(1-\gamma)(1-\beta)}{w_2-d_2} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) - \frac{3(2\alpha+1)\mathrm{B}\mu\big(1-\gamma+(1-\beta)(w_2-d_2)\big)^2}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} & if\ \mu \le \mu_0, \\ \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} - \frac{4(1-\gamma)^2}{(w_2-d_2)^2} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) + \frac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\,\mathrm{B}} & if\ \mu_0 \le \mu \le \mu_1, \\ \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} - \frac{2(1-\gamma)^2}{(w_2-d_2)^2} + \frac{2(1-\gamma)}{w_3-d_3} + 2(1-\beta) & if\ \mu_1 \le \mu \le \mu_2, \\ -\frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)} - \frac{4(1-\gamma)(1-\beta)}{w_2-d_2} - \frac{2(1-\gamma)}{w_3-d_3} - 2(1-\beta) + \frac{3(2\alpha+1)\mathrm{B}\mu\big(1-\gamma+(1-\beta)(w_2-d_2)\big)^2}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} & if\ \mu_2 \le \mu, \end{cases}$$

Where

$$\mu_0 = \frac{\mu_1(1-\gamma)}{1-\gamma+(1-\beta)(w_2-d_2)},$$

$$\mu_1 = \frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}},$$

$$\mu_2 = \frac{\mu_0(w_2-d_2)^2}{1-\gamma+(1-\beta)(w_2-d_2)}\left(\frac{4(1-\gamma)d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2}{w_3-d_3}+\frac{2(1-\beta)}{(1-\gamma)}+\frac{2(1-\beta)}{(w_2-d_2)}-\frac{(1-\gamma)}{(w_2-d_2)^2}\right).$$

Proof

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2) = 3(2\alpha+1)\mathrm{B}a_3 - 3(2\alpha+1)\mathrm{B}\mu a_2^2 . \quad (20)$$

From (18) we get, $a_2 = \frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}}+\frac{b_2}{2(\alpha+1)\mathrm{A}} \Rightarrow a_2^2 = \left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}}+\frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2$,
from (19) and the value a_2^2 substituting by
together in (20),

$$3(2\alpha+1)\mathrm{B}(a_3-\mu a_2^2) = (1-\beta)(c_2+b_2c_1)+b_3 - 3(2\alpha+1)\mathrm{B}\mu\left(\frac{c_1(1-\beta)}{2(\alpha+1)\mathrm{A}}+\frac{b_2}{2(\alpha+1)\mathrm{A}}\right)^2,$$

$$= (1-\beta)c_2+(1-\beta)b_2c_1+b_3 - 3(2\alpha+1)\mathrm{B}\mu\left(\frac{{c_1}^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}+\frac{c_1b_2(1-\beta)}{2(\alpha+1)^2\mathrm{A}^2}+\frac{{b_2}^2}{4(\alpha+1)^2\mathrm{A}^2}\right),$$

$$= (1-\beta)c_2+(1-\beta)b_2c_1+b_3 - \frac{3(2\alpha+1)\mathrm{B}\mu{c_1}^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}-\frac{3(2\alpha+1)\mathrm{B}\mu c_1b_2(1-\beta)}{2(\alpha+1)^2\mathrm{A}^2}-\frac{3(2\alpha+1)\mathrm{B}\mu{b_2}^2}{4(\alpha+1)^2\mathrm{A}^2},$$

$$= b_3-\frac{3(2\alpha+1)\mathrm{B}\mu{b_2}^2}{4(\alpha+1)^2\mathrm{A}^2}+(1-\beta)c_2+b_2c_1(1-\beta)\left(1-\frac{3(2\alpha+1)\mathrm{B}\mu}{2(\alpha+1)^2\mathrm{A}^2}\right)+{c_1}^2(1-\beta)^2\left(\frac{2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mathrm{B}\mu}{4(\alpha+1)^2\mathrm{A}^2}-\frac{1}{2}\right), \quad (21)$$

From (21), we have,

$$\begin{aligned}3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| &\le \left|b_3-\frac{3(2\alpha+1)\mathrm{B}\mu b_2^2}{4(\alpha+1)^2\mathrm{A}^2}\right|+\left|(1-\beta)c_2-\frac{1}{2}c_1^2(1-\beta)^2\right| \\ &+\frac{|c_1|^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}|2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}| \\ &+\frac{|b_2||(1-\beta)c_1|}{2(\alpha+1)^2\mathrm{A}^2}|2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}|. \quad (22)\end{aligned}$$

Now, consider the first case for all
$2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}\ge 0$

$$\Longrightarrow\therefore\ \mu \le \frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}$$

Note that
$2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\,\mathrm{B}>0$ and
$b_3-\frac{3(2\alpha+1)\mu b_2^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2}>0$, from lemma 1

$$\left|c_2(1-\beta)-\frac{c_1^2(1-\beta)^2}{2}\right| \le 2(1-\beta)-\frac{|c_1|^2(1-\beta)^2}{2}$$

and inequalities

$$|b_2| \le \frac{2(1-\gamma)}{(w_2-d_2)}$$

and

$$|b_3| \le \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3},$$

Substuting $|b_2|$, $|b_3|$ in (22)

$$\begin{aligned}3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| &\le \frac{4(1-\gamma)^2d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}-\frac{(4)3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{4(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}+2(1-\beta)-\frac{|c_1|^2(1-\beta)^2}{2} \\ &+\frac{2(1-\gamma)|c_1|(1-\beta)}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}[2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}] \\ &+\frac{|c_1|^2(1-\beta)^2}{4(\alpha+1)^2\mathrm{A}^2}[2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}],\end{aligned}$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}+2(1-\beta)-\frac{|c_1|^2(1-\beta)^2}{2} + \frac{2(1-\gamma)|c_1|(1-\beta)2(\alpha+1)^2\mathrm{A}^2}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}-\frac{2(1-\gamma)|c_1|(1-\beta)3(2\alpha+1)\mu\mathrm{B}}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2} + \frac{2|c_1|^2(1-\beta)^2(\alpha+1)^2\mathrm{A}^2}{4(\alpha+1)^2\mathrm{A}^2}-\frac{|c_1|^2(1-\beta)^2 3(2\alpha+1)\mu\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} -\frac{3(2\alpha+1)\mu|c_1|^2(1-\beta)^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2}+\frac{2(1-\gamma)|c_1|(1-\beta)(\alpha+1)^2\mathrm{A}^2}{2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}-\frac{3(2\alpha+1)(1-\gamma)(1-\beta)|c_1|\mu\mathrm{B}}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mathrm{B}\mu}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2}-\frac{3(2\alpha+1)\mu|c_1|^2(1-\beta)^2\mathrm{B}}{4(\alpha+1)^2\mathrm{A}^2} + \frac{(1-\gamma)(1-\beta)\left(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B}\right)|c_1|}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2} \quad (23)$$

$$= \mathcal{Q}(x)\,;x=\lceil c_1 \rceil.$$

We defined x

$$\mathcal{Q}(x)=x^2-\frac{4(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)\mu(1-\beta)(w_2-d_2)\mathrm{B}}x-\frac{16(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2 d_2}{3(w_3-d_3)(w_2-d_2)(2\alpha+1)(1-\beta)^2\mu\mathrm{B}} -\frac{8(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(w_3-d_3)(2\alpha+1)(1-\beta)^2\mu\mathrm{B}}-\frac{8(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(1-\beta)\mu\mathrm{B}}+\frac{4(1-\gamma)^2}{(w_2-d_2)^2(1-\beta)^2},$$

After doing some operations, we get

$$x=\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}$$

Now, substuting in (23)

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)\mu(1-\gamma)^2\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} -\frac{,3(2\alpha+1)\mu(1-\beta)^2\,\mathrm{B}\left(\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}\right)^2}{4(\alpha+1)^2\mathrm{A}^2} + \frac{(1-\gamma)(1-\beta)(\,2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})\left(\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}}\right)}{(w_2-d_2)(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} -\frac{4(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{(4)3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}}+\frac{2(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} +\frac{(1-\gamma)^2(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})^2}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} + \frac{(1-\gamma)^2\,(4(\alpha+1)^4\mathrm{A}^4-12(\alpha+1)^2(2\alpha+1)\mathrm{A}^2\mu\mathrm{B}+9(2\alpha+1)^2\mu^2\mathrm{B}^2)}{3(2\alpha+1)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2\mu\mathrm{B}},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2} + \frac{4(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{3(2\alpha+1)(1-\gamma)^2\mu\mathrm{B}}{(\alpha+1)^2(w_2-d_2)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{4(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}} - \frac{4(1-\gamma)^2}{(w_2-d_2)^2} \qquad (24)$$

Now $[x] \le 2$ we get interval

$$\frac{2(1-\gamma)(2(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\mu\mathrm{B})}{3(2\alpha+1)(w_2-d_2)(1-\beta)\mu\mathrm{B}} \le 2,$$

$$\frac{4(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{6(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} \le \mu\,,$$

then

$$\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} \le \mu \quad (25)$$

hence result (24) concludes for the case

$$\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)} \le \mu \quad \le \frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}$$

Second, consider the case $\mu \le \mu_0$

$$\mu \le \mu_0 = \frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)},$$

Write :

$$a_3-\mu a_2^2 = a_3-\mu_0 a_2^2+\mu_0 a_2^2-\mu a_2^2,$$

$$|a_3-\mu a_2^2| \le |a_3-\mu_0 a_2^2|+|\mu_0-\mu||a_2^2|, \qquad (26)$$

From lemma (3), we obtain:

$$|a_2| \le \frac{(1-\beta)}{(\alpha+1)\mathrm{A}}+\frac{(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}} \Rightarrow |a_2| \le \frac{(w_2-d_2)(1-\beta)+(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}} \qquad (27)$$

From (26), substituting $|a_3-\mu a_2^2|$,

Then, we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le 3(2\alpha+1)\mathrm{B}|a_3-\mu_0 a_2^2| + \quad 3(2\alpha+1)\mathrm{B}|\mu_0-\mu||a_2^2|, \qquad (28)$$

From (26) substituting $|a_3-\mu a_2^2|$, and from (24) substituting $3(2\alpha+1)\mathrm{B}\ |a_3-\mu_0 a_2^2|$, and from (27) substituting $|a_2^2|$,

and, we have

$$|\mu_0-\mu| = \left|\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}-\mu\right|,$$

$$= \left|\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}-\frac{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\right|,$$

because Positive $0 \le \alpha < 1 <\ ,0 \le \beta < 1$,

$0 \le \gamma < 1 <\ , w_2 \ge d_2$,

$$|\mu_0-\mu| = \frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}} \quad ,(29)$$

Now substituting in (28), we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \le \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mu\mathrm{B}} + 3(2\alpha+1)\mathrm{B}\left(\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mathrm{B}}\right)\left(\frac{(w_2-d_2)(1-\beta)+(1-\gamma)}{(w_2-d_2)(\alpha+1)\mathrm{A}}\right)^2,$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+ \quad 2(1-\beta) + \quad \frac{4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mathrm{B}}\left(\frac{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2}\right)- \quad \frac{4(1-\gamma)^2}{(w_2-d_2)^2} + \left(\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2-3(2\alpha+1)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)\mu\mathrm{B}}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}\right)\big((w_2-d_2)(1-\ \beta) + (1-\gamma)\big),$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{2(1-\gamma)\big((w_2-d_2)(1-\ \beta)+(1-\gamma)\big)}{(w_2-d_2)^2} - \frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{2(1-\gamma)(\alpha+1)^2\mathrm{A}^2\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2} - \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2 \leq| \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta) + \frac{2(1-\gamma)\big((w_2-d_2)(1-\ \beta)+(1-\gamma)\big)}{(w_2-d_2)^2}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2} + \frac{2(1-\gamma)\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)}{(w_2-d_2)^2}-\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}+\frac{2(1-\gamma)(w_2-d_2)(1-\ \beta)}{(w_2-d_2)^2} - \frac{4(1-\gamma)^2}{(w_2-d_2)^2}+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}+\frac{2(1-\gamma)(w_2-d_2)(1-\beta)}{(w_2-d_2)^2} - \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{4(1-\gamma)(1-\ \beta)}{(w_2-d_2)} - \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}. \quad 30)$$

Consider

$$\mu=\mu_1=\frac{2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}}, \qquad (31)$$

subsutating in (24), we get

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{4(1-\gamma)^2}{(w_2-d_2)^2} + \frac{(3)4(\alpha+1)^2(1-\gamma)^2\mathrm{A}^2\ (2\alpha+1)\mathrm{B}}{(2)3(2\alpha+1)(w_2-d_2)^2\mathrm{B}\ (\alpha+1)^2\mathrm{A}^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)+\frac{2(1-\gamma)^2}{(w_2-d_2)^2}-\frac{4(1-\gamma)^2}{(w_2-d_2)^2},$$

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}. \qquad (32)$$

Now Case $\mu_2 < \mu$,

$$a_3-\mu a_2^2 = a_3-\mu_2 a_2^2+\mu_2 a_2^2-\mu a_2^2 = a_3-\mu_2 a_2^2+(\mu_2-\mu)a_2^2, \qquad (33)$$

$$|a_2|^2 \leq \frac{\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}$$

Defined condition $\mu_2 < \mu$,

$$\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2} \leq -\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}-\frac{2(1-\gamma)}{w_3-d_3}-2(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)} + \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

we but μ in the right tip:

$$\frac{8(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{4(1-\gamma)}{w_3-d_3}+4(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}+\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)} \leq \frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},$$

we divide all terms by a coefficient, we get

$$\frac{8(1-\gamma)^2(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2 d_2}{3(2\alpha+1)(w_2-d_2)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} - \frac{2(1-\gamma)^2(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)^2\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\gamma)(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_2-d_2)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} \leq \mu.$$

$$\frac{8(1-\gamma)^2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2 d_2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} - \frac{2(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{4(1-\gamma)(1-\beta)(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} \leq \mu.$$

Now subsutating in this inequality

$$3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \leq 3(2\alpha+1)\mathrm{B}|a_3-\mu_2 a_2^2| + 3(2\alpha+1)\mathrm{B}|\mu_2-\mu||a_2|^2,$$

From (31) and we multiply this $|\mu_2-\mu|$ in negative

$$3(2\alpha+1)\mathrm{B}\left|a_3-\mu a_2^2\right| \leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2} + 3(2\alpha+1)\mathrm{B}\Bigg(-\frac{8(1-\gamma)^2(w_2-d_2)(\alpha+1)^2\mathrm{A}^2 d_2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} - \frac{4(1-\gamma)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)(w_3-d_3)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} - \frac{4(1-\beta)(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} + \frac{2(1-\gamma)^2(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2} - \frac{4(1-\gamma)(1-\beta)(w_2-d_2)(\alpha+1)^2\mathrm{A}^2}{3(2\alpha+1)\mathrm{B}\big((1-\gamma)+(w_2-d_2)(1-\beta)\big)^2}+\mu\Bigg)\left(\frac{\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}\right),$$

we get

$$
\begin{aligned}
3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \\
\leq \frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}+\frac{2(1-\gamma)}{w_3-d_3}+2(1-\beta)-\frac{2(1-\gamma)^2}{(w_2-d_2)^2}-\frac{8(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)} \\
-\frac{4(1-\gamma)}{(w_3-d_3)}-4(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)}+\frac{2(1-\gamma)^2}{(w_2-d_2)^2} \\
+\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2},
\end{aligned}
$$

then

$$
\begin{aligned}
3(2\alpha+1)\mathrm{B}|a_3-\mu a_2^2| \\
\leq -\frac{4(1-\gamma)^2 d_2}{(w_3-d_3)(w_2-d_2)}-\frac{2(1-\gamma)}{w_3-d_3}-2(1-\beta)-\frac{4(1-\gamma)(1-\beta)}{(w_2-d_2)} \\
+\frac{3(2\alpha+1)\mu\mathrm{B}\big((1-\beta)(w_2-d_2)+(1-\gamma)\big)^2}{(w_2-d_2)^2(\alpha+1)^2\mathrm{A}^2}. \ \blacksquare \quad (34)
\end{aligned}
$$

The many works already done on analytic functions associated by derivative operator see these references [6-12].

conclusion

In this paper, we studied Fekete-szego inequality defined on the new class of analytic univalent functions in open unit disk by using a new generalized derivative operator and Hadamard product with a normalized analytic function.

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