Efficient Analytical Solutions for Linear and Nonlinear Difference Equations Using the Discrete Putzer Algorithm

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Abstract: The discrete Putzer algorithm is examined in this work as an effective technique for resolving linear and nonlinear difference equations. The approach simplifies calculations and offers analytical answers by utilizing matrix theory and eigenvalue analysis, particularly for higher-order and non-homogeneous systems. Its efficacy is demonstrated by examples such as Fibonacci sequences and population dynamics. The Cayley-Hamilton theorem is applied to increase the algorithm's usefulness and make it a potent tool for dynamic systems in both theoretical and practical settings.

Keywords: Discrete Putzer Algorithm, Difference Equations, Cayley-Hamilton Theorem, Eigenvalue Analysis, Analytical Solutions.

Introduction:

he sciences in the area of difference equations, the difference equations are important for simulating dynamic systems in general. They are frequently employed to show phenomena in signal processing, economics, population dynamics, and other disciplines that study discrete processes. The need to figure out effective methods is motivated by the fact that solving these equations analytically, particularly higher-order and non-homogeneous ones, frequently poses substantial computational hurdles. One effective method for overcoming these obstacles is the discrete Putzer algorithm.
This approach streamlines the calculation of solutions to linear and nonlinear difference equations, building on the fundamental Cayley-Hamilton theorem and matrix theory. The discrete Putzer algorithm uses eigenvalue analysis to expedite the process in contrast to conventional techniques, especially for autonomous systems of differential equations.
Because of this, it works very well in applications that need exact answers, such population modelling and sequence analysis, which includes Fibonacci sequences.
Studies have demonstrated that the algorithm's accuracy and robustness are improved by spectral radius analysis and eigenvalue characteristics [1-7]. For [4] extended Putzer's representation to compute analytic matrix functions using omega matrix calculus.
It is a vital tool for both theoretical and practical applications due to its computational efficiency in handling higher-order systems. Through thorough examples, this study will explore the algorithm's efficacy for developing approaches to addressing dynamic systems.

Many research studies are conducted in fields like engineering, applied mathematics, and t

The Putzer algorithm

Theorem 1: Putzer algorithm [6]
The Putzer algorithm provides an efficient solution to the linear difference equation $u(t+1) = Au(t)$ with initial vector u_0 is

$$u(t) = \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0 = A^t u_0$$

Where M_i are given by
$M_0 = I \implies M_i = (A - \lambda I) M_{i-1} (1 \le i \le n)$
and $c_i(t), (i = 1,2,\cdots,n)$ are uniquely determined by equation

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ \vdots \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ \vdots \\ c_n(t) \end{bmatrix}$$

with

$$\begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ \vdots \\ c_n(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Proof:- According to the **Cayley-Hamilton theorem**, if A is an $n \times n$ matrix, then any power of A (i.e., A^n) can be expressed as a linear combination of $I, A, A^2, \cdots, A^{n-1}$.
This means that all powers of A can be written in terms of these powers. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A, which may not be distinct, with each eigenvalue repeated according to its multiplicity.

$$\left.\begin{array}{l} M_0 = I, \quad (the\ identity\ matrix), \\ M_i = (A - \lambda_i I)M_{i-1}, for\ i = 1,2,\dots,n \end{array}\right\} \quad (1)$$

Since $M_0 = I$ this recursion will eventually yield $M_n = 0$, which is consistent with the Cayley-Hamilton theorem. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A then the characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0$$

By Cayley-Hamilton theorem

$$(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I) = 0$$

Now, definition (1) implies that each A^i is a linear combination of $M_0, M_1, \cdots, M_i$ for $(i = 0,1,2,\cdots,n-1)$

$$A^t = \sum_{i=0}^{n-1} c_{i+1}(t)M_i\ for\ t \geq 0$$

where $c_{i+1}(t)$ are to be determined.

Now, we can write,

$$A^{t+1} = A.A^t$$

$$\sum_{i=0}^{n-1} c_{i+1}(t)M_i = A\sum_{i=0}^{n-1} c_{i+1}(t)M_i \quad (2)$$

$$= \sum_{i=0}^{n-1} c_{i+1}(t)AM_i$$

By using (1), we get

$$= \sum_{i=0}^{n-1} c_{i+1}(t)[M_{i+1} + \lambda_{i+1}M_i]$$

$$= \sum_{i=0}^{n-1} c_{i+1}(t)M_i + \sum_{i=0}^{n-1} c_{i+1}(t)\lambda_{i+1}M_i \quad (3)$$

$$M_n = 0 = \sum_{i=0}^{n-1} c_{i+1}(t)M_{i+1}$$

Replace i by $i-1$ in first sum and use the fact that

$$= \sum_{i=1}^{n-1} c_i(t)M_i + \sum_{i=0}^{n-1} c_{i+1}(t)\lambda_{i+1}M_i$$

Equation (3) is satisfied if the $c_i(t)$, $(i = 1,2,\cdots n)$ are chosen to satisfy the system: (Equating the coefficients of $M_i's$, we get)

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ \vdots \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ \vdots \\ c_n(t) \end{bmatrix} \quad (4)$$

Since $A^0 = I = c_1(0)I + \cdots + c_n(0)M_{n-1}$, we must have

$$\begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ \vdots \\ c_n(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

By theorem, the initial value problems $(4) - (5)$ has a unique solution, hence the theorem.

Example1: Putzer algorithm

Solve $u(t+1) = Au(t)$, where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

To find eigenvalues consider characters equation $|A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$

$$\Rightarrow (1-\lambda)^2 + 1 \Rightarrow 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i,$$

are the eigenvalues of matrix. Now, we know that

$$M_0 = I$$

$$M_1 = A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - (1+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

Also $c_1(t)$ and $c_2(t)$ satisfies the equations

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} \text{ and } \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \end{bmatrix} = \begin{bmatrix} 1+i & 0 \\ 1 & 1-i \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

And $c_1(0) = 1$, $c_2(0) = 0$

$$c_1(t+1) = (1+i)c_1(t), \quad c_1(0) = 1 \longrightarrow (6)$$

$$c_2(t+1) = c_1(t) + (1-i)c_2(t), \quad c_2(0) = 0 \longrightarrow (7)$$

Solving (6), we get $c_1(t) = (1+i)^t$

Then, (7) becomes

$$c_2(t+1) = (1-i)c_2(t) + (1+i)^t, c_2(0) = 0$$

$$c_2(t+1) - (1-i)c_2(t) = (1+i)^t, \quad c_2(0) = 0$$

$$\big(E - (1-i)\big)c_2(t) = (1+i)^t$$

Using annihilator method

$$[E - (1+i)][E - (1-i)]c_2(t) = 0$$

$$\because [E - (1+i)](1+i)^t = 0 \Rightarrow (E - \lambda)\lambda^t = 0$$

$$c_2(t) = A(1-i)^t + B(1+i)^t$$

Substituting in equation to find value of, we get

$$B = \frac{1}{2i} = -\frac{i}{2}$$

This gives,

$$c_2(t) = A(1-i)^t + -\frac{i}{2}(1+i)^t$$

But,

$$c_2(0) = 0 \Rightarrow \text{A} = \frac{i}{2} = -\frac{i}{2}$$

Finally, the solution is

$$u(t) = (c_1(t)I + c_2(t)M_1)u_0$$

$$u(t) = (1+i)^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{i}{2}[(1-i)^t - (1+i)^t]\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

which is the required solution.

From (**Putzer algorithm**)

$$A^t = (c_1(t)I + c_2(t)M_1) = 2^{1/2}\begin{bmatrix} \cos\frac{\pi}{4}t & \sin\frac{\pi}{4}t \\ -\sin\frac{\pi}{4}t & \cos\frac{\pi}{4}t \end{bmatrix}$$

The discrete Putzer algorithm

Autonomous homogeneous linear systems of difference equation. Consider a first order linear discrete system of the form

$$x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1k}x_k(n)$$

$$x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2k}x_k(n)$$

$$\vdots$$

$$x_k(n+1) = a_{k1}x_1(n) + a_{k2}x_2(n) + \cdots + a_{kk}x_k(n)$$

We can write this in vector form as

$$X(n+1) = AX(n)$$

where $X(n) = (x_1(n),\ x_2(n)\ ,\cdots, x_k(n)\ )^T$ and $A = (a_{ij})$ is a $k \times k$ matrix.
Suppose that $X(0)$ is known. Then

$$X(1) = AX(0)$$
$$X(2) = AX(1) = A^2X(0)$$
$$X(3) = AX(2) = A^3X(0)$$

Continuing with this pattern, we see that

$$X(n) = A^nX(0)$$
$$x(n+1) = \begin{pmatrix} x_1(n+1) \\ \vdots \\ x_k(n+1) \end{pmatrix} = Ax(n)\ , x(0) = \begin{pmatrix} x_1(0) \\ \vdots \\ x_k(0) \end{pmatrix}$$

An algorithm for calculating the matrix is provided by the following theorem A^n

Theorem 2: The discrete Putzer Algorithm [3]
Let A any $k \times k$ matrix with Eigen values $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then

$$A^n = \sum_{J=1}^{k} u_J(n)M(J-1)$$

Where

$$u_1(n) = \lambda_1^n\ , u_J(n) = \sum_{i=0}^{n=1} \lambda_J^{n-1-i} u_{J-1}(i)$$

For the proof see [3, 6]
Example 2: Consider the system of difference equations

$$x_1(n+1) = x_1(n) - x_2(n)$$
$$x_2(n+1) = 2x_2(n)$$

with initial conditions $x_1(0) = 0$ and $x_2(0) = 1$
If $x(n) = \left(x_1(n)\ , x_2(n)\right)^T$ and $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, then we have

$$X(n+1) = AX(n)\ , X(0) = (0,1)^T$$

In order to determine the eigenvalues of A, we solve the equation

$$|A - \lambda I| = (1-\lambda)(2-\lambda) = 0$$

The eigenvalues are $\lambda_1 = 1\ , \lambda_2 = 2$. Now from the Putzer algorithm
$A^n = \sum_{J=1}^{2} u_J(n)M(J-1) = u_1(n)M(0) + u_2(n)M(1)$,
Where $M(0) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$M(1) = A - \lambda_1 I = A - I = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Now, $u_1(n) = \lambda_1^n = 1^n = 1\ , u_2(n) = \sum_{i=0}^{n=1} \lambda_2^{n-1-i} u_1(i)$

$$= \sum_{i=0}^{n-1} 2^{n-1-i} = 2^{n-1} \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i = 2^{n-1} \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$
$$= 2^n - 1$$
$$\therefore\ A^n = 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (2^n - 1)\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 1 + 2^n - 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}$$

Hence the analytical solution is

$$X(n) = A^nX(0) = \begin{pmatrix} 1 & 1 - 2^n \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 2^n \\ 2^n \end{pmatrix}$$
$$\therefore\ x_1(n) = 1 - 2^n \ and\ x_2(n) = 2^n$$

By using MATLAB Code, we get Figure 1

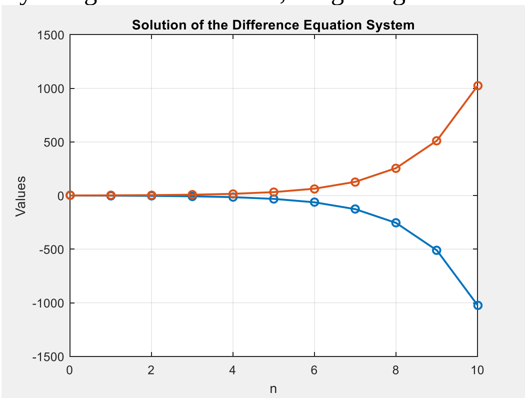

Figure1: Numerical Solution of the system of Differential Equations

Remark 1: The analysis and solution of any k-th order difference problem can be simplified by transforming it into a set of difference equations of the first order. This approach reduces the complexity of higher-order problems, particularly for numerical methods, by decomposing them into multiple first-order equations.
Example 3: To determine the analytical solution to the third-order system, apply the discrete Putzer method.

$$x(n+2) = 3x(n+2) - 3x(n+1) + x(n)$$
$$x(0) = 1\ , x(1) = 0\ , x(2) = 3$$

To solve this problem by writing the discrete system above as follows:

$$x(n+1) = y(n)$$
$$y(n+1) = z(n)$$
$$z(n+1) = 3z(n) - 3y(n) + x(n)$$

Therefore, the system of the first order linear difference equation as

$$X(n+1) = AX(n)$$

Where

$$X(n) = \left(x(n), y(n), z(n)\right)^T$$
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$$

The corresponding initial condition is,
$X(0) = \left(x(0), y(0), z(0)\right)^T = \left(x(0), y(1), z(2)\right)^T$
$= (1,0,3)^T$
To find the eigenvalue,

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix}$$
$$= -\lambda[(\lambda - 3) + 3] - 1(-1) =$$
$$= -\lambda(\lambda^2 - 3\lambda - 3) + 1 = -(\lambda - 1)^3 = 0$$

Therefore, eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 1$.
Now, from the Putzer algorithm,

$$A^n = \sum_{J=1}^{3} u_J(n)M_{(J-1)}$$
$$= M(0)u_1(n) + M(1)u_2(n) + M(2)u_3(n)$$

where, $M(0) = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$M(1) = A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix},$$

$$M(2) = (A-I)^2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$u_1(n) = \lambda_1^n = 1^n = 1$$

$$u_2(n) = \sum_{i=0}^{n-1} \lambda_2^{n-1-i} u_1(i) = \sum_{i=0}^{n-1} 1^{n-1-i}.1$$

$$= \sum_{i=0}^{n-1} 1 = n$$

$$u_3(n) = \sum_{i=0}^{n-1} \lambda_3^{n-1-i} u_2(i) = \sum_{i=0}^{n-1} 1^{n-1-i}.i$$

$$= \sum_{i=0}^{n-1} i = \frac{n\,(n-1)}{2}$$

Therefore,

$$A^n = M(0)u_1(n) + M(1)u_2(n) + M(2)u_3(n)$$

$$= 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix} + \frac{n(n-1)}{2}\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

Now, $X(n) = A^n X(0)$

$$= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{pmatrix} + \frac{n(n-1)}{2}\begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$x(n) = 1 - n + 2n(n-1) = 1 - n + 2n^2 - 2n$$
$$= 1 - 3n + 2n^2$$
$$y(n) = 0 + 3n + 2n(n-1) = 3n + 2n^2 - 2n$$
$$= n + 2n^2$$
$$z(n) = 3 + 7n + 2n(n-1) = 3 + 7n + 2n^2 - 2n$$
$$= 3 + 5n + 2n^2$$

It is easy to plot the numerical solution by using ode 45, we obtain Figure 2

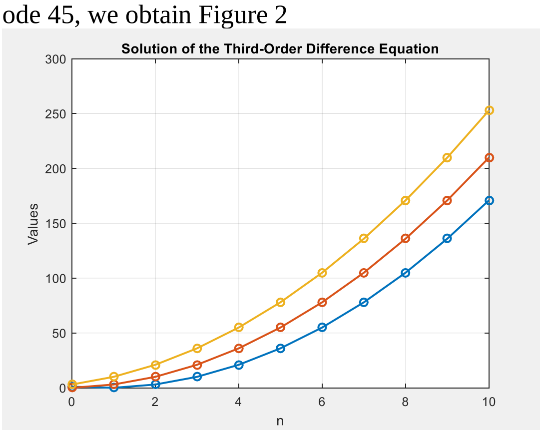

Figure 2: Numerical solution of the Third DEs

Remark2:

- The values for $x(n)$, $x(n+1)$, and $x(n+2)$ Every time step is printed.
- The plot shows how these values evolve over the steps from $n = 0$ to $n = 10$.

Non-Homogeneous systems of linear difference equations.

Consider the non-homogeneous linear system

$$X(n+1) = AX(n) + b(n)$$

Where $X(n)$ and $b(n)$ are $k \times 1$ column matrices and A is a $k \times k$. Then the analytical solution is given by

$$X(n) = A^n X(0) + \sum_{r=0}^{n-1} A^{n-r-1} b(r)$$

Where:

$X(0)$ is the initial state at $n = 0$. The first term, $A^n X(0)$, symbolises the remedy for the system's homogenous component. The second term, $\sum_{r=0}^{n-1} A^{n-r-1} b(r)$, accounts for the contribution of the non-homogeneous part.

The solution comprises two components: the homogeneous solution and the non-homogeneous solution. The term $A^n X(0)$ describes the system's evolution from the initial condition $X(0)$ over n steps, governed by the matrix A. Meanwhile, the summation term accounts for the non-homogeneous influences at each time step, incorporating the cumulative effect of the forcing function $b(n)$ across previous time steps.

Example 4: Use the discrete Putzer algorithm to find the analytical solution to the system

$$X(n+1) = \begin{pmatrix} 2 & 2 \\ -2 & 6 \end{pmatrix} X(n) + \begin{pmatrix} e^{-n} \\ 0 \end{pmatrix}, x(0) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

Solution:-

Let

$$A = \begin{pmatrix} 2 & 2 \\ -2 & 6 \end{pmatrix} \Rightarrow |A - \lambda I| = 0$$
$$\Rightarrow (\lambda - 2)(\lambda - 6) + 4$$
$$= \lambda^2 - 8\lambda + 12 + 4$$
$$= \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

Therefore, the eigenvalue of A are $\lambda_1 = 4 = \lambda_2$.

Now, from the Putzer algorithm.

$$A^n = \sum_{J=1}^{2} u_J(n) M(J-1)$$
$$= u_1(n)M(0) + u_2(n)M(1)$$

Where

$$M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M(1) = A - 4I = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix},$$

$$u_1(n) = \lambda_1^n = 4^n,$$

$$u_2(n) = \sum_{i=0}^{n-1} \lambda_2^{n-1-i} u_1(i) = \sum_{i=0}^{n-1} 4_2^{n-1-i} 4^i$$

$$= \sum_{i=0}^{n-1} 4^{n-1} = n4^{n-1}$$

$$\therefore\; A^n = u_1(n)M(0) + u_2(n)M(1)$$

$$= 4^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n4^{n-1}\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 4^n - 2n4^{n-1} & 2n4^{n-1} \\ -2n4^{n-1} & 4^n + 2n4^{n-1} \end{pmatrix}$$

Hence, the analytical solution is given by

$$\sum_{r=0}^{n-1}\begin{pmatrix}4^{n-r-1}-2(n-r-1)4^{n-r-2} & 2(n-r-1)4^{n-r-2}\\ -2(n-r-1)4^{n-r-2} & 4^{n-r-1}+2(n-r-1)\end{pmatrix}$$

$$=\begin{pmatrix}4^n-2n4^{n-1}+7(2n4^{n-1})\\ -2n4^{n-1}+7(4^n+2n)^{n-1}\end{pmatrix}$$

$$+\sum_{r=0}^{n-1}e^{-r}\begin{pmatrix}4^{n-r-1}-2(n-r-1)4^{n-r-2}\\ -2(n-r-1)4^{n-r-2}\end{pmatrix}=$$

$$=\begin{pmatrix}4^n+12n4^{n-1}\\ 7(4^n)+12n4^{n-1}\end{pmatrix}$$

$$+\sum_{r=0}^{n-1}e^{-r}\begin{pmatrix}4^{n-r-1}-2(n-r-1)4^{n-r-2}\\ -2(n-r-1)4^{n-r-2}\end{pmatrix}$$

By using numerical solution, we obtain Figure 4

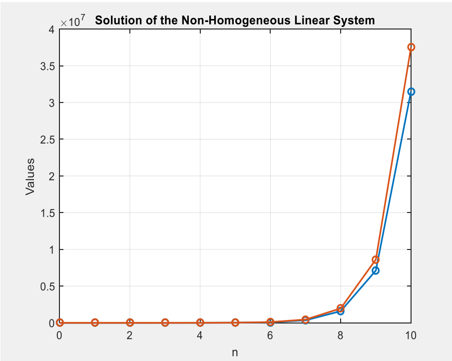

Figure 4: Solution of the Non-Homogeneous Linear System

Initial value problems for linear systems

Consider value problems for linear form

$$u_1(t+1)=a_{11}(t)u_1(t)+\cdots+a_{1n}(t)u_n(t)+f_1(t)$$
$$u_2(t+1)=a_{21}(t)u_1(t)+\cdots+a_{2n}(t)u_n(t)+f_2(t)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$u_n(t+1)=a_{n1}(t)u_1(t)+\cdots+a_{nn}(t)u_n(t)+f_n(t)$$

For, $t=a,a+1,a+2,\cdots$. This system can be written as an equivalent vector equation,

$$u(t+1)=A(t)u(t)+f(t) \qquad (8)$$

Where

$$u(t)=\begin{bmatrix}u_1(t)\\u_2(t)\\ \vdots\\u_n(t)\end{bmatrix},$$

$$A(t)=\begin{bmatrix}a_{11}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & \cdots & a_{21}(t)\\ \vdots & \vdots & \vdots\\ a_{n1}(t) & \cdots & a_{nn}(t)\end{bmatrix}_{n\times n},$$

$$f(t)=\begin{bmatrix}f_1(t)\\f_2(t)\\ \vdots\\f_n(t)\end{bmatrix}$$

The study of equation (8) includes the n^{th} order scalar equation

$$P_n(t)y(t+n)+\cdots+P_0(t)y(t)=r(t) \qquad (9)$$

As special case. To see this, let $y(t)$ solve equation (9) and define

$$u_i(t)=y(t+i-1) \quad \text{for} \quad 1\le i\le n\,, t=a,a+1,\cdots$$

Then the vector function $u(t)$ with components $u_i(t)$ satisfies equation (8) if

$$A(t)=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 1\\ \frac{-P_0(t)}{P_n(t)} & \frac{-P_1(t)}{P_n(t)} & \frac{-P_2(t)}{P_n(t)} & \cdots & \frac{-P_{n-1}(t)}{P_n(t)}\end{bmatrix}$$

$$,f(t)=\begin{bmatrix}0\\0\\0\\ \vdots\\ \frac{r(t)}{P_n(t)}\end{bmatrix} \qquad (10)$$

The matrix $A(t)$ in equation (10) is" called the companion matrix" of equation (9). From (9)

$$y(t+n)=\frac{r(t)}{P_n(t)}-\frac{-P_{n-1}(t)}{P_n(t)}y(t+n-1)-\frac{-P_{n-2}(t)}{P_n(t)}y(t+n-2)\cdots\frac{-P_0(t)}{P_n(t)}y(t)$$

Since,

$$u_1(t+1)=y(t+1)$$
$$u_2(t+1)=y(t+2)$$
$$u_3(t+1)=y(t+3)$$
$$\vdots$$
$$y_n(t+1)=y(t+n)$$

But

$$u_1(t)=y(t)$$
$$u_2(t)=y(t+1)$$
$$u_3(t)=y(t+2)$$
$$\vdots$$
$$y_n(t)=y(t+n-1)$$
$$\Rightarrow u_1(t+1)=u_2(t)\ ,u_2(t+1)=u_3(t)\cdots\cdots u_{n-1}(t+1)=u_n(t)$$

$$\Rightarrow u_n(t)=-\frac{P_0(t)}{P_n(t)}u_1(\mathrm{t})-\frac{P_1(t)}{P_n(t)}u_2(\mathrm{t})\cdots\cdots-\frac{P_{n-1}(t)}{P_n(t)}u_n(\mathrm{t})+\frac{r(t)}{P_n(t)}$$

Conversely, if $u(t)$ solves equation (8) with $A(t)$ and $f(t)$ given in equation (10), then $y(t)=u_1(t)$ is a solution of equation (9).

Theorem 3: [3] For each t_0 in $\{a,a+1,\cdots\}$ and each $n-vector\ u_0$. Equation (8) has a unique solution $u(t)$ defined for $t=t_0,t_0+1,\cdots$, so that $u(t_0)=u_0$.

Now assume that A is independent of t (i,e, all coefficients in the system are constants) and $f(t)=0$. Then the equation (8) reduces to

$$u(t+1)=Au(t) \qquad (11)$$

Then the solution $u(t)$ of equation (11) satisfying the initial condition

$u(0)=u_0\ , is\ u(t)=A^t u_0(t=0,1,2,\cdots)$.

Hence the solutions of equation (11) can be found by calculating powers of A.

Example 5: Population of American bison

Let $u_1(t), u_2(t)$ and $u_3(t)$ represent the number of calves, yearlings, and adults, respectively, after t years, and represent the American bison population vector.

Let's say that the number of infants is 42 percent of the number of adults from the year before. Assume additionally that 75% of yearlings grow into adults, 95% of adults survive to live the next year, and 60% of calves live to become yearlings each year. The linear system is therefore satisfied by the population vector $u(t)$.

$$u_1(t+1) = 0.42u_3(t)$$
$$u_2(t+1) = 0.60u_1(t)$$
$$u_3(t+1) = 0.75u_2(t) + 0.95u_3(t)$$

(i,e) $u(t+1) = Au(t)$, where

$$A = \begin{bmatrix} 0 & 0 & 0.42 \\ 0.60 & 0 & 0 \\ 0 & 0.75 & 0.95 \end{bmatrix}_{3\times3}, u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$

To solve this problem by using also ode 45, we get the following figure.

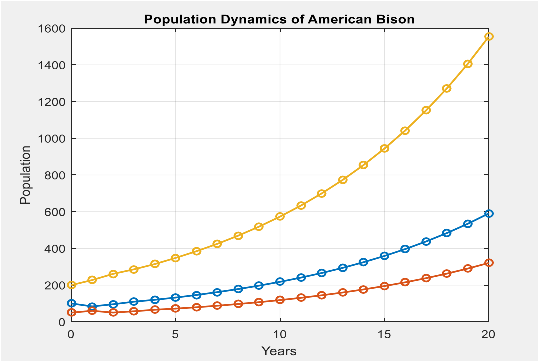

Figure 5: Population Dynamics of American Bison Graphs showing the changes in the populations of calves, yearlings, and adults over the years.

Conclusion

We have demonstrated the effectiveness of analytical and computational approaches, such as matrix-based solutions and the Putzer Algorithm, in solving both homogeneous and non-homogeneous systems. The use of matrix exponentiation provides an efficient method for solving linear homogeneous problems, while non-homogeneous solutions are obtained as the sum of the homogeneous response and a specific solution derived through iterative summations. These techniques have practical applications in fields such as population dynamics and robotic motion analysis.

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