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Fuzzy Mathematics
(Fuzzy Sets)

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Abstract:

In this paper, we explore the aspects of important branch of fuzzy mathematics, by utilizing fuzzy theory. The field of fuzzy mathematics has been making rapid progress in recent years. Motivated by the practical success of fuzzy mathematics in many other branches, there has been an increasing amount of work on the rigorous theoretical studies of fuzzy sets. Firstly, we start this paper by introduction about branches of fuzzy theory and the history of fuzzy mathematics. In the second section of this paper we introduce the concept fuzzy sets. Additionally, a comparison between classical set theory and fuzzy set theory. Finally, we clarify the basic concepts associated with fuzzy set and study the operations on it.

Keywords: fuzzy mathematics, fuzzy sets.

الرياضيات الضبابية

(المجموعات الضبابية)

الملخص :

في هذه الورقة، نحن نستكشف جوانب فرع مهم من الرياضيات الضبابية، من خلال استخدام النظرية الضبابية. أحرز مجال الرياضيات الضبابية تقدماً سريعاً في السنوات الأخيرة. بدافع النجاح العملي للرياضيات الضبابية في العديد من الفروع الأخرى، كان هناك قدر متزايد من العمل على الدراسات النظرية الدقيقة للمجموعات الضبابية. أولاً، نبدأ هذه الورقة بمقدمة حول فروع النظرية الضبابية وتاريخ

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الرياضيات الضبابية. في القسم الثاني من هذه الورقة نقدم مفهوم المجموعات الضبابية. بالإضافة إلى ذلك، مقارنة بين نظرية المجموعة الكلاسيكية ونظرية المجموعة الضبابية. أخيراً، نوضح المفاهيم الأساسية المرتبطة بالمجموعة الضبابية ودراسة العمليات عليها.

الكلمات المفتاحية: الرياضيات الضبابية، المجموعات الضبابية.

Introduction:

By fuzzy theory we mean all the theories that use the basic concept of fuzzy set or continuous membership function. Fuzzy theory can be roughly classified according to Fig.(1). There are five major branches: (i) fuzzy mathematics, where classical mathematical concepts are extended by replacing classical sets with fuzzy sets; (ii) fuzzy logic and artificial intelligence, where approximations to classical logic are introduced and expert systems are developed based on fuzzy information and approximate reasoning; (iii) fuzzy systems, which include fuzzy control and fuzzy approaches in signal processing and communications; (iv) uncertainty and information, where different kinds of uncertainties are analyzed; and (v) fuzzy decision making, which considers optimalization problems with soft constraints.

Of course, these five branches are not independent and there are strong interconnections among them. For example, fuzzy control uses concepts from fuzzy mathematics and fuzzy logic.

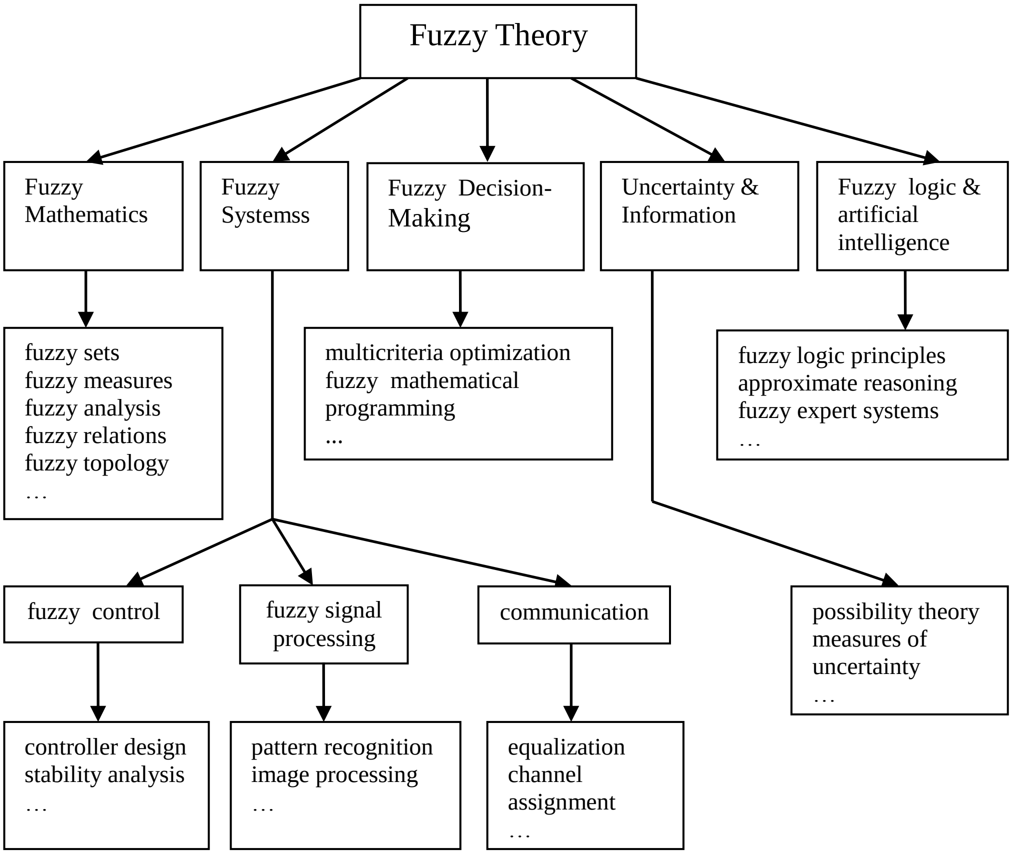

Figure (1) Classification of fuzzy theory.

Fuzzy mathematics forms a branch of mathematics including fuzzy set theory and fuzzy logic. It started in 1965 after the publication of Lotfi Asker Zadeh's seminal work fuzzy sets. (Lotfi Zadeh presented the concept of fuzzy sets as a generalization of normal sets, and it gives a more accurate description of natural phenomena instead of the description given by ordinary set theory, and since then scientists have tended to apply the concept of fuzzy sets in most branches of theoretical and applied mathematics. This extended to all other sciences such as computer science, life sciences, economics, geography,... etc.)

Fuzzy mathematics provide the starting point and basic language for many other branches of fuzzy theory. Fuzzy mathematics by itself is a huge field, where fuzzy mathematical principles are developed by replacing the sets in classical mathematical theory with fuzzy sets. In this way, all the classical mathematical branches may be" fuzzified." We have seen the birth of fuzzy measure theory, fuzzy topology, fuzzy algebra, fuzzy analysis, etc. Understandably, only a small portion of fuzzy mathematics has found applications in engineering.

From Classical sets to Fuzzy Sets.

Let U be the universe of discourse, or universeal set, which contains all the possible elements of concern in each particular context or application. Recall that a classical (crisp) set A, or simply a set A, in the universe of discourse U can be defined by listing all of its members (the list method) or by specifying the properties that must be satisfied by the members of the set (the rule method). The list method can be used only for finite sets and is therefore of limited use. The rule method is more general. In the rule method, a set A is represented as

$$A = \{ x \in U | x \text{ meets some conditions} \} \quad (1)$$

There is yet a third method to define a set A −the membership method, which introduces a zero-one membership function (also called characteristic function, crimination function, or indicator function) for A, denoted by $\mu_A(x)$, such that

$$\mu_A(x) = \begin{cases} 1 & if \;\; x \in A \\ 0 & if \;\; x \notin A \end{cases} \quad (2)$$

The set A is mathematically equivalent to its membership function $\mu_A(x)$ in the sense that knowing $\mu_A(x)$ is the same as knowing A itself.

Definition. A fuzzy set in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval [0,1].

Therefore, a fuzzy set is a generalization of a classical set by allowing the membership function to take any values in the interval [0,1]. In other words, the membership function of a classical set can only take two values zero and one, whereas the membership function of a fuzzy set is a continuous function with range [0,1] We see from the definition that there is nothing "fuzzy " about a fuzzy set; it is simply a set with a continuous membership function.

A fuzzy set A in U may be represented as a set of ordered pairs of a generic element x and its membership values, that is,

$$A = \{ (x, \mu_A(x)) | \; x \in U \} \quad (3)$$

When U is continuous (for example, $U = R$), A is commonly written as

$$A = \int_U \mu_A(x) \, /x \quad (4)$$

where the integral sign does not denote integration; it denotes the collection of all points $x \in U$ with the associated membership function $\mu_A(x)$.

$$A = \sum_U \mu_A(x) \, /x \quad (5)$$

where the summation sign does not represent arithmetic addition; it denotes the collection of all points $x \in U$ with the associated membership function $\mu_A(x)$.

Classical set theory and fuzzy sets.

- A set is any well defined collection of objects.
- An object in a set is called an element or member of that set.
- Sets are defined by a simple statement.

- Describing whether a particular element having a certain property belongs to that particular set.

$$A = \{a_1, a_2, a_3, \dots \dots, a_n\} \qquad (6)$$

- If the elements a_i $(i = 1, 2, 3, \dots \dots, n)$ of a set A are subset of universal set X, then set A can be represented for all elements $x \in X$ by its characteristics function

$$\mu_A(x) = 1 \text{ if } x \in X \text{ otherwise } 0 \qquad (7)$$

- Fuzzy sets theory is an extension of classical set theory.
- Fuzzy sets is fully defined by its membership functions.
- Membership function is a function in [0,1] that represents the degree of belonging.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set.

A comparison between classical set theory and fuzzy set theory.

classical set theory	**fuzzy set theory**
♦ classes of objects with sharp boundaries. ♦ A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries. ♦ Widely used in digital system design.	♦ classes of objects with un-sharp boundaries. ♦ A fuzzy set defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries. ♦ Used in fuzzy controllers.

We now consider the next example of fuzzy sets and from it draw some remarks.

Example. Let Z be a fuzzy set named “numbers close to zero.” Find the possible membership functions for Z ?

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Solution: (1) First a possible membership function for Z is

$$\mu_z(x) = e^{-x^2} \tag{8}$$

where $x \in R$. This is a Gaussian function with mean equal to zero and standard derivation equal to one. According to this membership function, the numbers 0 and 2 belong to the fuzzy set Z to the degrees of $e^0 = 1$ and e^{-4} , respectively.

(2) We also may define the membership function for Z as

$$\mu_z(x) = \begin{cases} 0 & if \quad x < -1 \\ x+1 & if \quad -1 \le x < 0 \\ 1-x & if \quad 0 \le x < 1 \\ 0 & if \quad 1 \le x \end{cases} \tag{9}$$

According to this membership function, the numbers 0 and 2 belong to the fuzzy set Z to the degrees of 1 and 0, respectively.(8) and (9) are plotted graphically in Figs. (2) and (3), respectively. We can choose many other membership functions to characterize “numbers close to zero”.

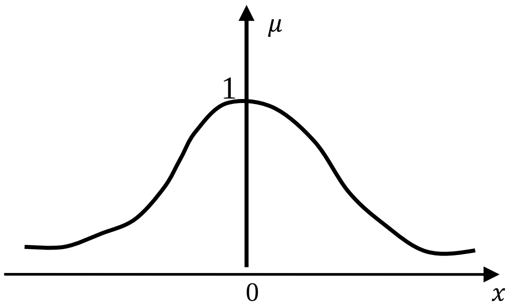
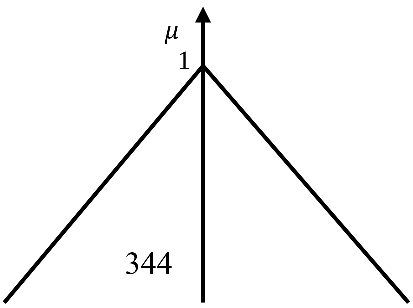


Figure (2) A possible membership function to characterize “numbers close to zero.”



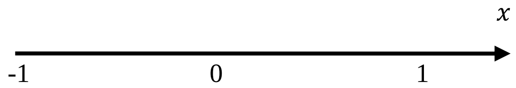


Figure (3) Another possible membership function to characterize "numbers close to zero."

Basic Concepts Associated with Fuzzy Set.

We now introduce some basic concepts and terminology associated with a fuzzy set.

Many of them are extensions of the basic concepts of a classical (crisp) set, but some are unique to the fuzzy set framework.

Definition. The concepts of support, fuzzy singleton, center, crossover point, height, normal fuzzy set, α -cut, convex fuzzy set, and projections are defined as follows.

- The support of a fuzzy set A in the universe of discourse U is a crisp set that contains all the elements of U that have nonzero membership values in A, that is, where supp (A) denotes the support of fuzzy set A. For example, the support of fuzzy set "several" in Fig. (4) is the set of integers $\{3,4,5,6,7,8\}$. If the support of a fuzzy set is empty, it is called an empty fuzzy set.
- A fuzzy singleton is a fuzzy set whose support is a single point in U.

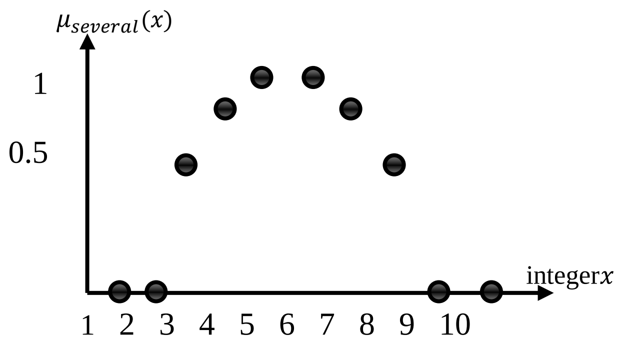


Figure (4) Membership function for fuzzy set "several"

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- The center of a fuzzy set is defined as follows: if the mean value of all points at which the membership function of the fuzzy set achieves its maximum value is finite, then define this mean value as the center of the fuzzy set; if the mean value equals positive (negative) infinite, then the center is defined as the smallest (largest) among all points that achieve maximum membership value. Fig (5) shows the centers of some typical fuzzy set.
- The crossover point of fuzzy set is the point in U whose membership value in A equals 0.5.
- The height of a fuzzy set is the largest membership value attained by any point. For example, the heights of the fuzzy set in Fig.(3) equal one. if the height of a fuzzy set equal one, it is called a normal fuzzy set. The fuzzy set in Figs.(3) is therefore normal fuzzy sets.
- An α -cut of a fuzzy set A is a crisp set A_α that contains all the elements in U that have membership values in A greater than or equal to α, that is,

$$A_\alpha = \{ x \in U \,|\mu_A (x) \geq \alpha\} \tag{10}$$

For example, for $\alpha = 0.3$, the $\alpha =$ cut of the fuzzy set (9) (Fig. 3) is the crisp set [-0.7,0.7], and for $\alpha = 0.9$, it is [-0.1,0.1].

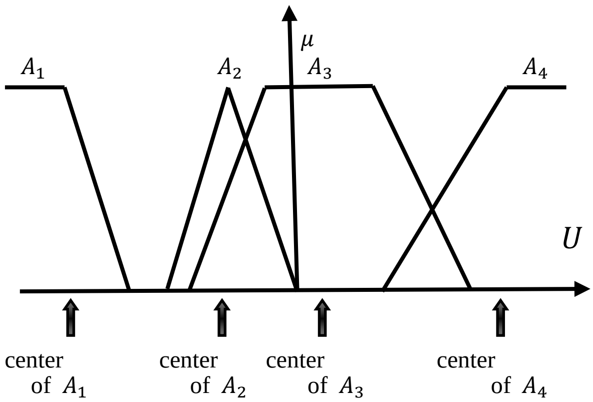

Figure (5) Centers of some typical fuzzy sets.

Operations on Fuzzy Sets.

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The basic concepts introduced in previous parts concern only a single fuzzy set. In this part, we study the basic operations on fuzzy set. In the sequel, we assume that A and B are fuzzy sets defined in the same universe of discourse U.

Definition. The equality, containment, complement, union, and intersection of two fuzzy sets A and B are defined as follows.

Equality: $A = B \iff \mu_A(x) = \mu_B(x)$ for all $x \in U$ (11)

Containment: $A \subset B \iff \mu_A(x) \leq \mu_B(x)$ for all $x \in U$ (12)

Complement: $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$ (13)

Union: $\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$ (14)

Intersection: $\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)]$ (15)

The reader may wonder why we use "max" for union and "min" for intersection; we now give an intuitive explanation. An intuitively appealing way of defining the union is the following: the union of A and B is the smallest fuzzy set containing both A and B. More precisely, if C is any fuzzy set contains both A and B, then it also contains the union of A and B. To show that this intuitively appealing definition is equivalent to (14), we note, first, that $A \cup B$ as defined by (14) contains both A and B because $\max[\mu_A, \mu_B] \geq \mu_A$ and $\max[\mu_A, \mu_B] \geq \mu_B$.

Therefore, $\mu_c \geq \max[\mu_A, \mu_B] = \mu_{A\cup B,}$ which means that $A \cup B$ as defined by (14) is the smallest fuzzy set containing both A and B. The intersection as defined by (15) can be justified in the same manner.

Example. Consider the two fuzzy sets F and G defined in the interval $U = [0, 10]$ by the membership functions

$$\mu_F(x) = \frac{x}{x+2} \quad , \quad \mu_G(x) = 2^{-x} \tag{16}$$

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Determine the mathematical formulas of membership functions of each of the following fuzzy sets:

(a) $\overline{F}, \overline{G}$

(d) $F \cup G\ , F \cap G$

Solution: (a) $\overline{F} \rightarrow \mu_{\overline{F}}(x) = 1 - \mu_F(x) = 1 - \frac{x}{x+2} = \frac{2}{x+2}$

$$\overline{G} \rightarrow \mu_{\overline{G}}(x) = 1 - \mu_G(x) = 1 - 2^{-x}$$

(b) $F \cup G \rightarrow \mu_{F \cup G}(x) = max[\,\mu_F(x), \mu_G\,(x)]$

$$= max\left[\frac{x}{x+2},\ 2^{-x}\right] = \frac{x}{x+2}$$

$$F \cap G \rightarrow \mu_{F \cap G}(x) = min\,[\,\mu_F(x), \mu_G\,(x)]$$

$$= min\left[\frac{x}{x+2},\ 2^{-x}\right] = 2^{-x}$$

Example. Consider the two sets

$$A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\} \qquad (17)$$

$$B = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\} \qquad (18)$$

defined in the interval $U = [0, 1]$

Find the value of the following membership functions:

$\mu_{A \cap B}(x_1)$, $\mu_{A \cap B}(x_2)$, $\mu_{A \cap B}(x_3)$

Solution:

$$A \cap B = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$$

$$\mu_{A \cap B}(x_1) = \min\big(\mu_A(x_1),\ \mu_B(x_1)\big) = \max(0.5, 0.8) = 0.5$$

$$\mu_{A \cap B}(x_2) = 0.2\ \ and\ \ \mu_{A \cap B}(x_3) = 0$$

Example. Consider the set

$$A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\} \tag{19}$$

defined in the interval $U = [0, 1]$

Find the value of the following membership functions:

$\mu_{\overline{A}}(x_1)$, $\mu_{\overline{A}}(x_2)$, $\mu_{\overline{A}}(x_3)$

Solution:

$$\overline{A} = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1)\}$$

$$\mu_{\overline{A}}(x_1) = 1 - \mu_A(x_1) = 1 - 0.5 = 0.5$$
$$\mu_{\overline{A}}(x_2) = 0.3 \ \ and \ \ \mu_{\overline{A}}(x_3) = 1$$

As an examples, let us consider the following lemma.

Lemma. The De Morgan's Laws are true for fuzzy sets. That is, suppose A and B are fuzzy sets**,** then

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{20}$$

and

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{21}$$

Proof: First, we will prove (17).

We show that the following identity is true:

$$1 - max[\mu_A, \mu_B] = min[1 - \mu_A, 1 - \mu_B] \tag{22}$$

To show this we consider the two possible cases: $\mu_A \geq \mu_B$ and $\mu_A < \mu_B$. If $\mu_A \geq \mu_B$, then $1 - \mu_A \leq 1 - \mu_B$ and $1 - max[\mu_A, \mu_B] = 1 - \mu_A = min[1 - \mu_A, 1 - \mu_B]$, which is (22) If $\mu_A < \mu_B$, then $1 - \mu_A > 1 - \mu_B$ and $1 - max[\mu_A, \mu_B] = 1 - \mu_B = min[1 - \mu_A, 1 - \mu_B]$, which is again (22). Hence, (22) is true. From the definitions (13)-(15) and the definition of the equality of two fuzzy sets, we see that (22) implies (20).

Second, we will prove (21).

We show that the following identity is true:

$$1 - min[\,\mu_A, \mu_B\,] = max\,[\,1 - \mu_A, 1 - \mu_B\,] \qquad (23)$$

To show this we consider the two possible cases: $\mu_A \le \mu_B$ and $\mu_A > \mu_B$. If $\mu_A \le \mu_B$, then $1 - \mu_A \ge 1 - \mu_B$ and $1 - mim[\,\mu_A, \mu_B\,] = 1 - \mu_B = max\,[1 - \mu_A, 1 - \mu_B\,]$, which is (23) If $\mu_A > \mu_B$, then $1 - \mu_A < 1 - \mu_B$ and $1 - min[\,\mu_A, \mu_B\,] = 1 - \mu_B = max\,[\,1 - \mu_A, 1 - \mu_B\,]$, which is again (23). Hence, (23) is true. From the definitions (13)-(15) and the definition of the equality of two fuzzy sets, we see that (23) implies (21).

Conclusion

In this paper, we have dealt with one of the branches of fuzzy mathematics, which is fuzzy sets. Fuzzy sets are considered one of the most important branches of fuzzy mathematics, as all branches of fuzzy theory also depend on it. In this paper, we study the concept of the fuzzy set and also the concept associated with it, in addition to the operations on it.

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