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Solving the viscous Burger's equation using three methods

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Abstract:

Burger's equation is one of the very few nonlinear partial differential equations, which can be solved exactly using transformations for arbitrary initial and boundary conditions. So we consider the analytical solution of Burger's equation using three different methods, which are: the Cole-Hopf transformation, variational iteration method and Adomian decomposition method. And provide some examples to describe how we can solve this equation under different conditions. .

Keywords: Burger's equations, Cole-Hopf transformation, variational iteration method; Adomian decomposition method.

حل معادلة Burger باستخدام ثلاث طرق مختلفة

الملخص:

معادلة Burger تعتبر من احدى المعادلات التفاضلية الجزئية غير الخطية، التي يمكن حلها عن طريق تطبيق التحويلات ذات الشروط الحدية والابتدائية. لذلك في هذه الورقة نعرض كيفية حل معادلة Burger بواسطة ثلاث طرق مختلفة وهي: طريقة تحويلات كول هوبف، طريقة التكرار المتغير و طريقة تحليل أدوميان، مع أمثلة توضح كيفية حل هذه المعادلة تحت شروط مختلفة.

الكلمات المفتاحية: معادلة Burger، تحويل كولف هوب، طريقة التكرار المتغير، طريقة تحليل ادوميان.

1. Introduction

In the past several decades, many authors mainly had paid attention to burger's equation that first appeared in a 1915 by Bateman [4] who used the equation as a model for the motion of a viscous fluid when the viscosity approaches zero and derive two types of steady state solutions for the infinite domain problem. More than thirty years later, Johannes M. Burgers [6] introduced the equation in his attempt to formulate a simple mathematical model that would show the fundamental features exhibited by the turbulence in the hydrodynamic flows. Hence the name Burgers equation was given to this equation. The Diffuseness equations are Initial Value Problems which are dependent on time in the event Time Dependent Equations, the solution for this type of issues to be in an open area R which are subject to the terms of the border, this issues produced in the study of pressure waves in the fluid and in the proliferation of effort and displacement and the spread of Heat, etc., and in general that access to Analytical Solution to Diffuseness Problems are often complex in spite of the growing sophistication in concepts and methods sports used to solve nonlinear partial differential equations, so the Numerical Solutions is the best way to study the properties of this type of equations [12]. So that, studying the properties of the Burger's equation is necessary for the fluid mechanics equations Common. Cole and Hopf ([7],[10]) independently discovered that the Burger's equation with an initial condition on the infinite domain could be reduced to the linear heat equation by using change in variables.

Burger's equation is one of the very few nonlinear partial differential equations which can be solved exactly owing a transformation for arbitrary initial and boundary condition. However it is well known that exact solution of Burger's equation can be computed for restricted values of kinematic viscosityεas illustrated in the study of Miller [ijams, 2010; 180-182]. Thus numerical methods are opted for small values ofε. This equation is a useful test case of numerical methods due to its simplicity and predictable dynamics. The challenge is to resolve the sharp gradient or shocks that occur at small and vanishing viscosity and accurately tract their evolution. Some researches applied a finite element method constructed on the method of descretization in time to get a good approximate numerical solution of Burger’s equation. The numerical solution of Burger’s equation is of great importance due to the equation’s application in the approximate theory of flow through a shock wave travelling in a viscous fluid and in the Burger’s model of turbulence. It is solved analytically for arbitrary initial conditions [2].

The aim of this paper is how to solve Burger’s equation by using three different methods.

2. The governing equation [3]

The Burger’s equation is considered one of the fundamental model equations in fluid mechanics. The equation demonstrates the coupling between diffusion and convection processes.

The standard form of Burger’s equation is given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}=0,\; t>0, \qquad (1)$$

where ε is a constant that defines the kinematic viscosity. If the viscosity $=0$, the equation is called inviscidBurger’s equation. The inviscid Burger’s equation governs gas dynamics. The inviscid Burgers equation has been discussed before as a homogeneous case of the advection problem. Nonlinear Burger’s equation is considered by most as a simple nonlinear partial differential equation incorporating both convection and diffusion in fluid dynamics. Burger introduced this equation in to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission.

3. Analysis of methods

1- The Cole-Hopf transformation [7] [10][13]

Hopf and Cole discovered the Hopf-Cole transformation independently around 1950. By using this method we can transform the non-linear equation to an easy solved linear equation.

Theorem 1

if $\phi(x,t) > 0$ is any positive solution to the linear heat equation
$\phi_x = v\phi_{xx}$, then;

$$u(x,t) = \frac{\partial}{\partial x}\left(-2vlog\ v(t,x)\right) = -2v\frac{\phi_x}{\phi}$$

Solve Burger's equation

$$u_t + uu_x = vu_{xx}.$$

Which the Cole-Hopf transformation gives in:

$$u = -2v\left(\frac{\phi_x}{\phi}\right),$$

2- Variational iteration method [2]

To illustrate its basic concepts of the variational iteration method, we consider the following differential equation:

$$Lu + Nu = g(x)\ ,$$

where L is a linear operator, N a nonlinear operator, and $g(x)$,an inhomogeneous term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) - \int_0^x \lambda\left(\mathrm{Lu_n}(\tau) + \mathrm{N\breve{u}_n}(\tau) - \mathrm{g}(\tau)\right)d\tau\ , \qquad (2)$$

Where λ is a general Lagrangian multiplier [3], which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation,$\mathrm{\breve{u}_n}$is considered as a restricted variation[3], [5] i.e.

$$\mathrm{\breve{u}_n} = 0,$$

3- Adomian decomposition method [9]

We begin with the equation

$$Lu + R(u) + N(u) = g(t), \qquad (3)$$

where L is the operator of the highest-ordered derivatives and R is theremainder of the linear operator. The nonlinear term is represented by$N(u)$.

Thus we get

$$Lu = g(t) - R(u) - N(u), \qquad (4)$$

The inverse,

$$L^{-1} = \int_0^t (\cdot)dt, \quad (5)$$

operating with the operator L^{-1} on both sides of Eq. (1) we have

$$u = f_0 + L^{-1}\big(g(t) - R(u) - N(u)\big), \quad (6)$$

where f_0 is the solution of homogeneous equation

$$Lu = 0, \quad (7)$$

involving the constants of integration. The integration constants involved in the solution of homogeneous equation (7) involve the constants of integration. The integration constants involved in the solution of homogeneous equation (7) are to be determined by the initial or boundary condition accordingly as the problem is initial-value problem or boundary-value problem. The Adomian decomposition method assume that the unknown function $u(x,t)$can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (8)$$

and the nonlinear operator $N(u)$can be decomposed by an infinite series ofpolynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \quad (9)$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomialsof $u_0, u_1, \dots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda}\left[F\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2, \dots \quad (10)$$

Theorem 2

The solution of the nonlinear PDEs in the form (3) with the initial$u(x,0) = f(x)$ can be determined by the series (8) with the iterative

$$u_0(x,t) = f(x),$$

$$u_{n+1}(x,t) = f(x) + L^{-1}(R(u_n) - A_n), \;\; n \geq 0, \quad (11)$$

4. Applications

4.1 Solve Burger's equation by Cole-Hopf transformation [3]

Burger's equation (1) can also be written as

$$u_t + \left(\frac{1}{2}u^2 - vu_x\right)_x = 0, \tag{12}$$

where we have replaced the viscosity coefficient ε by v. Let

$$u = \psi_x, \tag{13}$$

and

$$vu_x - \frac{1}{2}u^2 = \psi_t, \tag{14}$$

Using equation (13) in equation (14) we get

$$v\psi_{xx} - \frac{1}{2}{\psi_x}^2 = \psi_t, \tag{15}$$

Putting$\psi = -2v \log \phi$log ϕso that

$$u = \psi_x = -2v\frac{\phi_x}{\phi}, \tag{16}$$

we get

$$\psi_{xx} = 2v\left(\frac{\phi_x}{\phi}\right)^2 - \frac{2v}{\phi}\phi_{xx}\,, \tag{17}$$

and

$$\psi_t = -2v\frac{\phi_t}{\phi}\,, \tag{18}$$

Using equations (16), (17), (18) in equation (4) we get

$$\phi_t = v\phi_{xx}\,, \tag{19}$$

which is the linear diffusion equation whose solution are well known. Substituting back on the known solution ofϕwe get the solution of Burger's equation. Equation (16) is known as Hopf-Cole Transformation. We see that this transformation eliminates the nonlinear term of Burger's equation. So equation (19) is also known as Linear Burger's equation. This will lead to exact solutions, each solution depends on the given conditions.

4.2 Solve Burger's equation by Variational iteration method [2]

Consider that the Burger's equation has the form

$$u_t + uu_x - \nu u_{xx} = 0, \tag{20}$$

with an initial condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)\exp(\gamma)}{1 + \exp(\gamma)}, \quad t \geq 0, \tag{21}$$

Where$\gamma = (\alpha/\nu)(x - \lambda)$and the parameters$\alpha, \beta, \lambda,$ and$\nu$are arbitrary constants.To solve Eq. (20) by means of the variational iteration method, we construct a correction functional which reads

$$u_{n+1}(x,t) = u_n(x,0) + \int_0^t \lambda\{u_t + u\tilde{u}_x - \nu\tilde{u}_{xx}\}d\tau, \tag{22}$$

Where $\delta\tilde{u}_n$is considered as a restricted variation. Its stationary conditions can be obtained as follows:

$$\lambda'(\tau) = 0, \tag{23a}$$

$$1 + \lambda(\tau)]_{\tau=t} = 0, \tag{23b}$$

Eq. (23a) is called Lagrange–Euler equation, and Eq. (23b) natural boundary condition. The Lagrange multiplier, therefore, can be identified as $\lambda = -1$, and the following variational iterationformula can be obtained:

$$u_{n+1}(x,t) = u_n(x,0) + \int_0^t \lambda\{(u_t)_n + u_n u_{nx} - \nu u_{nxx}\}d\tau, \tag{24}$$

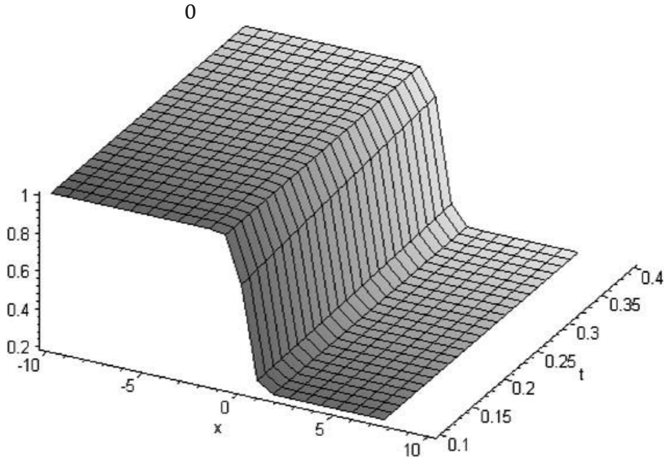

Figure 1

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Fig. 1. The behavior of $u(x,t)$evaluates by variational iteration method versus$x$for different values of time with fixed values$\nu = 1, \varepsilon = 1, \lambda = 0.125, \beta = 0.6, \alpha = 0.4$.

We start with an initial approximation$u_0 = u(x,t)$ given by Eq. (23), by the above iteration formula(7), we can obtain directly the other components as

$$u_1(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} + \frac{2\alpha\beta^2\exp(\gamma)}{\nu[1+\exp(\gamma)]^2}t, \qquad (25)$$

$$u_2(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\gamma)}{1+\exp(\gamma)} + \frac{2\alpha\beta^2\exp(\gamma)}{\nu[1+\exp(\gamma)]^2} + \frac{\alpha^3\beta^2\exp(\gamma)[-1+\exp(\gamma)]}{\nu^2[1+\exp(\gamma)]^3}t^2, \qquad (26)$$

$$u_3(x,t) = u_2 + \frac{\alpha^4\beta^3\exp(\gamma)[1-4\exp(\gamma)+\exp(\gamma)^2]}{3\nu^3[1+\exp(\gamma)]^4}t^3, \qquad (27)$$

and so on, in the same manner the rest of components of the iteration formula (24) were obtained using the Maple Package. The solution of $u(x,t)$in a closed form is

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)\exp(\zeta)}{1+\exp(\zeta)}, \qquad (28)$$

Where $\zeta = (\alpha/\nu)(x-\beta t-\lambda)$, which are exactly the same as obtained by Adomian decomposition method. The behavior of the solutions obtained by the variational iteration method is shown for different values of times in Fig. 1.

4.3 Solve Burger's equation by Adomain decomposition method [3]

We consider the Burger's equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = f(x), \qquad (29)$$

Applying the inverse operator L_t^{-1} to (1) leads to

$$u(x,t) = f(x) + L_t^{-1}(u_{xx}) - L_t^{-1}(uu_x), \tag{30}$$

Using the decomposition series for the linear term $u(x,t)$ and the series of Adomian polynomials for the nonlinear termuu_xgive

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L_t^{-1}\left((\sum_{n=0}^{\infty} u_n(x,t))_{xx} \right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n \right), \tag{31}$$

Identifying the zeroth component $u_0(x,t)$ by the term that arise from the initial conditionand following the decomposition method, we obtain the recursive relation

$$\begin{aligned} &u_0(x,t) = f(x), \\ &u_{k+1}(x,t) = L_t^{-1}\left(u_{k_{xx}}\right) - L_t^{-1}(A_k), \;\; k \geq 0, \end{aligned} \tag{32}$$

The Adomian polynomials for the nonlinear term uu_xhave been derived in the form

$$\begin{aligned} A_0 &= u_{0_x} u_0, \\ A_1 &= u_{0_x} u_1 + u_{1_x} u_0, \\ A_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\ A_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0, \\ A_4 &= u_{0_x} u_4 + u_{1_x} u_3 + u_{2_x} u_2 + u_{3_x} u_1 + u_{4_x} u_0, \end{aligned} \tag{33}$$

In view of (32) and (33), the first few components can be identified by

$$\begin{aligned} u_0(x,t) &= f(x), \\ u_1(x,t) &= L_t^{-1}\left(u_{0_{xx}}\right) - L_t^{-1} A_0, \\ u_2(x,t) &= L_t^{-1}\left(u_{1_{xx}}\right) - L_t^{-1} A_1, \\ u_3(x,t) &= L_t^{-1}\left(u_{2_{xx}}\right) - L_t^{-1} A_2, \end{aligned} \tag{34}$$

Additional components can be elegantly computed to enhance the accuracy level. The solution in a series form follows immediately, however, the *n*-term approximant $\phi_n$can be determined by

$$\phi_n = \sum_{k=0}^{n-1} u_k(x,t), \tag{35}$$

Example1

Consider the Burger’s equation:

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \tag{36}$$

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With the initial condition

$$u(x,0) = \sin \pi x, \tag{37}$$

and the homogeneous boundary conditions

$$u(x,0) = 0,$$
$$u(1,0) = 0, \qquad t > 0,$$

By the Hopf-Cole transformation

$$u(x,0) = -2v\frac{\theta_x}{\theta}, \tag{38}$$

The Burger's equation transforms to the linear heat equation

$$\frac{\partial\theta}{\partial t} = v\frac{\partial^2\theta}{\partial x^2}, \tag{39}$$

However, by the Hopf-Cole transformations, initial condition (37) and the above mentioned homogeneous boundary condition transform following conditions (40) and (41), respectively;

$$\theta_0(x) = \theta(x,0) = e^{-(2v\pi)^{-1}[1-\cos(\pi x)]}, \qquad 0 < x < 1, \tag{40}$$

$$\theta_x(0,t) = \theta_x(1,t) = 0, \qquad t > 0, \tag{41}$$

where if $\theta = \theta(x,t)$ is any solution of heat equation (39) is a solution of Burger's equation (36) with the initial condition (37) and the above boundary conditions. Hence, using the method of separation of variables the Fourier series solution to the above problem by equations (39)-(41) can be obtained easily as

$$\theta(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 vt}\cos(n\pi x), \tag{42}$$

where $a_0$and $(n = 1,2,\dots)$ are Fourier coefficients and they are evaluated inthe usual manner

$$a_0 = \int_0^1 e^{-(2v\pi)^{-1}[1-\cos(\pi x)]}dx, \tag{43}$$

$$a_n = 2\int_0^1 e^{-(2v\pi)^{-1}[1-\cos(\pi x)]}\cos(n\pi x)\,dx, \quad (n = 1,2,\dots), \tag{44}$$

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Thus, using the Hopf-Cole transformation given by equation (38), the exact Fourier solution to the problem given by equations (36),(37) and the abovementioned boundary conditions are obtained as

$$u(x,t)=\frac{2\pi v\sum_{n=1}^{\infty}a_n e^{-n^2\pi^2 vt}n\sin(n\pi x)}{a_0\sum_{n=1}^{\infty}a_n e^{-n^2\pi^2 vt}\cos(n\pi x)}\,, \quad (45)$$

Example2

Consider the following initial value equation

$$\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}\,,$$
$$u(x,0)=2x,\quad t>0,$$

For this example, we use the Variational iteration method iterative formula (2), can be written as

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial \mathrm{u_n}}{\partial \xi}+\mathrm{u_n}\frac{\partial \mathrm{u_n}}{\partial \mathrm{x}}-\frac{\partial^2 \mathrm{u_n}}{\partial \mathrm{x}^2}\right)d\xi\,, \quad (46)$$

Starting with$\mathrm{u_0(x,t)=2x}$, the following results can be derived from iterative formula (46),

$$\mathrm{u_1(x,t)=2x-4xt},$$
$$\mathrm{u_2(x,t)=2x-4xt+8xt^2-\frac{16}{3}xt^3},$$
$$\mathrm{u_3(x,t)=2x-4xt+8xt^2-16xt^4+\frac{64}{3}xt^4-\frac{64}{3}xt^5+\frac{128}{9}xt^6-\frac{256}{63}xt^7},$$
$$\mathrm{u_4(x,t)=2x-4xt+8xt^2-16xt^4+32xt^4-\frac{832}{15}xt^5+\frac{256}{3}xt^6}$$
$$\mathrm{-\frac{7424}{63}xt^7+\frac{9088}{63}xt^8-\frac{88064}{567}xt^9+\frac{45056}{315}xt^{10}}$$
$$\mathrm{-\frac{20480}{189}xt^{11}+\frac{4096}{63}xt^{12}-\frac{16384}{567}xt^{13}+\frac{32768}{3969}xt^{14}}$$
$$\mathrm{-\frac{65536}{59536}xt^{15}},$$
$$\mathrm{u_4(x,t)=2x-4xt+8xt^2-16xt^4+32xt^4-64xt^5+\frac{5504}{45}xt^6}$$
$$\mathrm{-\frac{3328}{15}xt^7+\frac{120704}{315}xt^8+\cdots},$$
$$\vdots$$

Thus, we have:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),$$
$$= 2x(1-2t+4t^2-8t^3+16t^4-\cdots),$$
$$= \sum_{n=0}^{\infty}(-1)^n 2^{n+1} x t^n,$$
$$= \frac{2x}{1+2t},$$

This is an exact solution.

Example 3

Solve the following Burger's equation:

$$u_t + uu_x = u_{xx}\,,$$
$$u(x,0) = 1-\frac{2}{x}\,, \quad x>0,$$

Solution.

The decomposition method proceeding as before gives

$$\sum_{n=0}^{\infty} u_n(x,t) = 1-\frac{2}{x} + L_t^{-1}\left(\left(\sum_{n=0}^{\infty} u_n(x,t)\right)_{xx}\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right),$$

Consequently, we set the recursive relation

$$u_0(x,t) = 1-\frac{2}{x},$$
$$u_{k+1}(x,t) = L_t^{-1}\big(u_{k_{xx}}(x,t)\big) - L_t^{-1}(A_k), \ \ k\geq 0,$$

that gives

$$u_0(x,t) = 1-\frac{2}{x},$$
$$u_1(x,t) = L_t^{-1}\big(u_{0_{xx}}(x,t)\big) - L_t^{-1}(A_0) = L_t^{-1}\left(-\frac{2}{x^2}\right) = -\frac{2}{x^2}t,$$
$$u_2(x,t) = L_t^{-1}\big(u_{1_{xx}}(x,t)\big) - L_t^{-1}(A_1) = L_t^{-1}\left(-\frac{4}{x^3}t\right) = -\frac{2}{x^3}t^2,$$
$$u_3(x,t) = L_t^{-1}\big(u_{2_{xx}}(x,t)\big) - L_t^{-1}(A_2) = L_t^{-1}\left(-\frac{6}{x^4}t^2\right) = -\frac{2}{x^4}t^3,$$

The series solution

$$u(x,t) = 1-\frac{2}{x}-\frac{2}{x^2}t-\frac{2}{x^3}t^2-\frac{2}{x^4}t^3+\cdots,$$

is readily obtained. To determine the exact solution, the last Eq. can be rewritten as

$$u(x,t) = 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \frac{t^3}{x^3} + \cdots\right) = 1 - \frac{2}{x}\left(\frac{1}{1-\frac{t}{x}}\right) = 1 - \frac{2}{x-t},$$

Conclusion

examples illustrate the effectiveness and simplicity of three methods. It's worth pointing out that the results of examples 2 and 3 are exactly the same as the result of applying Adomian decomposition method. As an advantage of variational iteration method over Adomian decomposition procedure, the former method provides the solution without calculating Adomian's polynomials. This technique solves the problem without any need for discretization of the variables.

Reference

[1] Abbasbandy S, & Darvishi M T, 2005, "*A numerical solution of Burgers equation by modified Adomian method"*, Appl. Math. Comp., 163, pp. 1265-1272.

[2] Abdou M. A. & Soliman A., 2005, "*Variational iteration method for solving Burger's and coupled Burger's equations"*, Journal of Computational and Applied Math. 181, pp 245-251.

[3] Abdul – Majid Wazwaz, 2009, *"Partial Differential Equation and Solitary Waves Theory***",** Bejing and Springer-Verlag Berlin Heidelberg, Chicago, IL 60655, USA.

[4] Bateman H, 1915, "*Some recent researches on the motion of fluids"*, Mon. Weath. Rev., 43, pp. 163-170.

[5] Biazar J. & Aminikhah H., 2009, "*Exact numerical solutions for non-linear Burger's equation by VIM*", Math. And Comp. Modellig 49, pp 1394-1400.

[6] Burgers J M, 1939, "*Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion"*, Trans. Roy. Neth. Acad. Sci., Amsterdam, 17, pp. 1-53.

[7] Cole J D, 1951, "*On a quasillinear parabolic equation occurring in Aerodynamics*", Quart. Appl. Math., 9, pp. 225-236.

[8] Darvishi M. T. & Javidi M, 2006, "*A numerical solution of Burgers equation by Pseudo Spectral method and Darvishi's preconditioning*", Appl. Math. Comp., 173, pp. 421-429.

[9] Hassan N.A., Ismail K. R., Aziza A. & Abdo R., 2004, "*Adomian decomposition method for Burger's-Huxley and Burger's-Fisher equations*", Applied Math. And Comp. 159, pp 291-301

[10] Hopf E., 1950, "*The Partial Differential Equation* $u_t + uu_x = vu_{xx}$", Comm. Pure Appl. Math., 3, pp. 201-230.

[11] Hou-de Han, 2006, "*Artificial boundary method for Burgers equation using nonlinear boundary conditions*", JCM, 24(3), pp. 295-304.

[12] Leonenko, N. & Meluikovn, 2001, *"Renarmahzation and Homogenization of the solutions of heat equation with linear Potential and related Burger's equation with random data",* Theory probab & Math. Statistics 1, PP.27-64.

[13] Milke Landajuela, 2011, "*Burgers equation*", BCAM Internship – Summer 2011.