Global Asymptotic Stability of Nonlinear Difference Equation

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Abstract: In this paper we obtain a global asymptotic stability of nonlinear difference equation of the form

$$y_{n+1} = y_n g(y_n), \qquad n = 0,1,2,....$$

Keywords: Difference equation, Global Attractivity, Equilibrium.

1. Introduction

Recently, many researchers are established the global attractive result for a linear and nonlinear difference equation. In [5] obtained necessary and sufficient conditions for the asymptotic stability of the difference equation

$$y_{n+1} - ay_n + by_{n-k} = 0, \qquad n = 0,1,2,.... \qquad (1)$$

Where the coefficients a and b are real numbers and k is a nonnegative integer,[6],[7],[8] investigated the global attractivity result for a second order nonlinear difference equation

$$y_{n+1} = y_n f(y_{n-1}), \qquad n = 0,1,2,.... \qquad (2)$$

established the global attractivity result for the nonlinear delay difference equation

$$y_{n+1} = y_n f(y_{n-k}), \qquad n = 0,1,2,.... \qquad (3)$$

where k is a nonnegative integer, and obtained a global attractivity result for the positive equilibrium of the nonlinear second order difference equation of the form

$$y_{n+1} = f(y_n, y_{n-1}), \qquad n = 0,1,2,.... \qquad (4)$$

[3] studied the global attractivity of the nonlinear difference equation

$$y_{n+1} = \frac{a+by_n}{A+y_{n-k}}, \qquad n = 0,1,2 ..... \qquad (5)$$

where $a, b, A \in (0, \infty), k$ is a positive integer and the initial conditions $y_{-k}, ..., y_{-1}$ and y_0 are arbitrary positive numbers, In the paper [1] the asymptotic behavior of the higher order nonlinear difference equation

$$y_{n+1} = ay_n + bg(y_n) + cg(y_{n-k}), \quad n = 0,1, ... \qquad (6)$$

is investigated, where a, b and c are constants with $0 < a < 1,\ 0 \le b < 1, 0 \le c < 1$ and $a + b + c = 1, g \in C\big[[0,\infty),[0,\infty)\big]$ with $g(y) > 0$ for $y > 0$ and k is a positive integer. A sufficient condition for the global attractivity of positive solution of eq (6) is obtained. Applications to some difference equation models are also given in [1], and in [2] studied the asymptotic behavior of the higher order nonlinear difference equation with unimodal terms

$$y_{n+1} = ay_n + by_n g(y_n) + cy_{n-k} g(y_{n-k}),$$
$$n = 0,1,2,.... \qquad (7)$$

where a, b and c are constants with

$$0 < a < 1, \quad 0 \le b < 1, 0 \le c < 1$$

And

$$a + b + c = 1, g \in C[[0,\infty),[0,\infty)]$$

is decreasing, and k is a positive integer, obtained some new sufficient conditions for the global attractivity of positive solutions of the equation. Our aim in this paper is the establish the following global asymptotic stability result for the nonlinear second order difference equation.

$$y_{n+1} = y_n g(y_n), \quad n = 0,1,2,.... \qquad (8)$$

Where the function g satisfies the following condition:

1. $g \in C[[0,\infty),[0,\infty)]$
2. $g(u)$ is non-increasing in u
3. The equation $g(y) = 1$ has a unique positive solution.
4. If y^* denotes the unique positive solution of $g(y) = 1$, then

$$(yg(y) - y^*)(y - y^*) > 0, \qquad y \ne y^*$$

y^* is a global attractor of all positive solution of eq (8).

By a solution of eq (8) we mean a sequence $\{y_n\}$ which is defined for $n \ge -1$ and which satisfies eq (8) for $n \ge 0$. if a_{-1} and a_0 are two given non negative numbers. then eq (8) has a unique solution satisfying the initial condition

$y_{-1} = a_{-1}$ and $y_0 = a_0$ (9)

If $a_{-1} > 0$ and $a_0 > 0$, then clearly for $n \geq 0$, the initial value problems eqs (8) and (9) have positive solutions.

In this paper, we will only investigate solution of eq (8) which are positive for $n \geq 0$. Such solution will also be called positive solution.

The discrete delay logistic model is an immediate applications of eq(8).

$$y_{n+1} = \frac{y_n}{b - a y_n}, \quad n = 0,1,2,\dots \tag{10}$$

Where $a \in (0,\infty)$, $b \in (1,\infty)$.

In general, most nonlinear difference equations cannot be solved directly.

However, some types of nonlinear equations can be solved. Usually are converted into a linear equation. We let

$$z_n = \frac{1}{y_n}$$

In eq(10) we give as

$$z_{n+1} - b z_n + a = 0 \tag{11}$$

Eq (11) is asymptotically stable iff all the roots of its characteristic equation.

$$\lambda^{k+1} - b\lambda^k + a = 0, \ k \text{ is apositive integer.} \tag{12}$$

Are inside the unit disk.

Theorem 1. Let a and b be arbitrary real numbers and k be a positive integer. then eq(12) is symptotic stability provided that

$$|a| + |b| < 1$$

Proof. [3]

Theorem 2. Suppose that eq(8) has a unique positive equilibrium y^* .then y^* is globally asymptotically stable.

Proof. Let y_n be a solution of eq (8) with initial conditions $y_0, y_{-1} \in (0,\infty)$,we must prove that y^* is stable and that

$$\lim_{n\to\infty} y_n = y^* \tag{13}$$

There are only two cases to consider:

Case one: for n sufficiently large either

$$y_n \geq y^* \tag{14}$$

or

$$y_n \leq y^* \tag{15}$$

We will assume eq(14) holds. the case where eq(15) hold is similar and will be left out ,then for n sufficiently large.

$$y_{n+1} \leq y_n g(y^*) = y_n$$

and so $\{y_n\}$ decreases to positive limit, $\lambda \geq y^*$. by taking limits on both sides of eq(8) we find $g(\lambda) = 1$ and in view of eq(9) ,$\lambda = y^*$.the proof of case one is complete.

Case two: Suppose that $\{y_n\}$ is strictly oscillatory solution. there exists a sequence of positive integers $\{\rho_i\}$ for $i = 1,2,3,\dots$ such that $k < \rho_1 < \rho_2 < \cdots$,

$\lim_{n\to\infty} \rho_n = \infty, \rho_n < y^*$

let $\{y_{\rho_i+1}, y_{\rho_i+2}, \dots, y_{\gamma_i}\}$ be the i −th negative semicycle of $\{y_n\}$ followed by the i-th positive semicycle $\{y_{\delta_i+1}, y_{\delta_i+2}, \dots, y_{\rho_i+1}\}$

for each $i = 1,2,\dots$. let y_{m_i} and y_{μ_i} be the minimum and maximum values in these two semicycles respectively ,with the smallest positive indices m_i and μ_i.it follows that

$$\mu_i - \gamma_i \leq 1 \text{ and } m_i - \rho_i \leq 1 \tag{16}$$

First, we consider the case where the maximum value in the positive semi cycle is equal to the first term of semi cycle. then $\mu_i - 1 = \gamma_i$ and

$$y_{m_i} \leq min\{y_{\mu_i-2}, y_{\mu_i-1}\} \leq max\{y_{\mu_i-2}, y_{\mu_i-1}\} \leq y^* \leq y_{\mu_i}$$

And

$$y_{\mu_i} = g(y_{\mu_i-1}, y_{\mu_i-2}) \leq g(y^*, y_{\mu_i-2}) < g(y^*, y_{m_i})$$

$g(y^*, y_{m_i}) > y^*$ and $g(g(y^*, y_{m_i}), y_{m_i}) > g(y^*, y_{m_i})$

Hence

$$y_{\mu_i} < g(y^*, y_{m_i}) g(g(y^*, y_{m_i}), y_{m_i}) = F(y_{m_i})$$

Next consider the case where the maximum value in the positive semi cycle is equal to second term of the semi cycle. Then

$$\mu_i - 2 = \gamma_i, \qquad y_{m_i} \leq y_{m_i-1} \ and$$

$$y_{m_i-2} < y^* \leq y_{m_i-1} < y_{m_i}$$

Also,

$$y_{\mu_i} = g(y_{m_i-1}, y_{m_i-2}) = g\left((y_{m_i-2}, y_{m_i-3}), y_{m_i-2}\right)$$
$$\leq g(g(y^*, y_{m_i-1}), y_{m_i-2}) \leq g(g(y^*, y_{m_i}), y_{m_i}) = F(y_{m_i})$$

In all cases that is

$$y_{\mu_i} \leq F(y_{m_i}) \tag{17}$$

from eq(17) it follows that

$$y_{\mu_i} < g(g(y^*, 0), 0) = A \tag{18}$$

By a parallel argument we have

$y_{m_i} > g(g(y^*, A), A) = B$ (19)

There exists an integer n_0 such that

$B < y_n < A \quad for \;\; n \geq n_0$ (20)

Where the constant A and B are defined by (18) and (19) respectively

Let

$$\left.\begin{array}{l}\lambda = \lim_{n\to\infty} inf(y_n) = \lim_{n\to\infty} inf\left(y_{m_i}\right) \\ ,\mu = \lim_{n\to\infty} sup\,\{y_n\} = \lim_{n\to\infty} sup\,\{y_{\mu_i}\}\end{array}\right\} \qquad (21)$$

The eq(20) exists are such that

$0 < B \leq \lambda \leq y^* \leq \mu \leq A$ (22)

We need show that

$\lambda = \mu = y^*$ (23)

If $\xi \in (0, \infty)$ and $\xi \in (0, \lambda)$ and from eq(21) ,then there exists $n_0 \in N$ such that

$\lambda - \xi \leq y^* \leq \mu + \xi \quad for \quad n \geq n_0 - 1$ (24)

From eq(17)we have

$y_{\mu_i} \leq g(\lambda - \xi)$

Which ξ is arbitrary,from eq(21) flows that

$\mu \leq G(\lambda)$ (25)

In the same way we find

$\lambda \geq G(\mu)$ (26)

That is mean G is decreasing and by eqs(22),(25)and (26) the proof is complete

. 3. Conclusions

. In this paper we gave the nonlinear second order difference equation (8) and established the global asymptotic stability of this equation.

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