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A comparison between Galerkin's and HaarWavelet Methods for Solving Linear Differential Equations

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Abstract: In this paper, two numerical methods are applied to solve linear differential equations and their performance is compared. These methods are the Galerkin method, a finite element method based on transforming the differential equation into an integral equation and then using basis functions to approximate the solution, and the Haar wavelet collocation method, which relies on using Haar wavelets as basis functions. The numerical results of both methods are compared with the exact solution to evaluate the accuracy and efficiency of each method.

Key words: linear differential equations , Galerkinmethod, Haar wavelet method.

مقارنة بين طريقتي جاليركين وهار المويجات لحل المعادلات التفاضلية الخطية

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الملخص: في هذه الورقة البحثية يتم تطبيق طريقتين عدديتين لحل المعادلات التفاضلية الخطية ومقارنة أدائهما، وهما: طريقة جاليركين وهي طريقة من طرق العناصر المنتهية، تعتمد على تحويل المعادلة التفاضلية إلى معادلة تكاملية، ومن ثم استخدام دوال أساسية لتقريب الحل، و طريقة مصفوفة العوامل لموجات هار وهي طريقة تعتمد على استخدام موجات هار كدوال أساسية، و يتم مقارنة النتائج العددية لكلا الطريقتين مع الحل الدقيق لتقييم دقة وكفاءة كل طريقة.

الكلمات المفتاحية : المعادلات التفاضلية الخطية،طريقة جاليركن ،طريقة مويجات هار .

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1. Introduction

Haar wavelets methods have become an increasingly popular tool in the computational sciences. They have been applied to awide range of problems, including signal analysis , data compression , and numerous other areas .Lepik(2006) had applied haar wavelet method to solve nonlinear integral equations. The high accuracy of the method was evident, even for a small number of collocation points. Lepik(2007) had applied Haar wavelet method to solve the Burgers and sine-Gordon equations, and the calculations showed that the accuracy of the solutions is very high even with a small number of grid points. and Change(2008) applied Haar wavelet for solving differential equation. The Galerkin method is one of the best known methods for finding numerical solutions of ordinary and partial differential equations .

differential equation has the form the

$$Lu = f \qquad (1)$$

With initial conditions :

$$u(0) = \alpha, \quad \acute{u}(0) = \beta, \quad \{\alpha, \beta\} \in \mathbb{R}$$

Where L is a differential operator defined on the real numbers set over the $L^2([0,1])$ space , which denotes the Hilbert space of square integrable functions on $[0,1]$ and f is a given function .

This paper aims to study the numerical solution for linear differential equations by applying Galerkin method and Haar wavelet method . The results obtained from these methods will be compared with exact solutions to assess their accuracy and efficiency.

2.Galerkin method "Kostadinova. J et .all (2007)"

Equation (1) will be solved by Galerkin method supposing that $\{\varphi_j\}$ is a basis functions for$L^2([0,1])$ and every φ_i satisfying C^2 on $[0,1]$ and $\mho$ is a some finite set of indices$j$ and consider the subspace

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$$S = span\{\varphi_j \ , j \in \mho\}$$

i.e the set of all finite linear combination of the elements $\{\varphi_j\}, j \in \mho$

The Galerkin technique searches on approximation $\tilde{u}$ of the exact solution u of the equation (1) in the form

$$\tilde{u} = \sum_{i \in \mho} a_i \, \varphi_i \quad \in s \qquad (2)$$

Where the coefficients a_i , $i \in \mho$ are unknown .

The residual is defined as

$$R(t) = L[\tilde{u}(t)] - f$$

Now , lets choose a weight function $w(x)$ such that

$$\langle w, R(t)\rangle = \langle w, L[\tilde{u}(t)] - f\rangle = \int_0^1 w(t)(L[\tilde{u}(t)] - f)dt = 0$$

If the weight function w is chosen from the basis function $\varphi_j \forall \ j \in \mho$, then

$$\langle w, R\rangle = \int_0^1 \varphi_j(L[\tilde{u}(t)] - f)dt = 0$$

These are sets of n-order linear equations , solve them , obtain all the coefficients and then we determine $\tilde{u}$ by equation (2)

Then ,we present the numerical results obtained by applying this method to examples.

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3. Haar wavelet method

3.1 Haar wavelet " N.Berwal et.all 2014"

Haar wavelets are created by integrating pairs of piecewise functions.

Furthermore , Haar functions are orthogonal, which makes them an excellent transform basis .

Suppose that the interval $[0,1]$ is portioned into 2^{J+1} subintervals of equal length, where$\Delta t = \frac{1-0}{2^{J+1}} = \frac{1}{2^{J+1}}$ $, J \in N$ determines the highest level of resolution . Haar wavelet family is defined

$$h_i(t) = \begin{cases} 1 & \frac{k}{2^j} \leq t < \frac{k+0.5}{2^j} \\ -1 & \frac{k+0.5}{2^j} \leq t < \frac{k+1}{2^j} \\ 0 & otherwise \end{cases} \tag{3}$$

Where $j = 0,1,2, \dots \dots, J \; and \; K = 0,1,2, \dots, 2^j - 1$, the index i is evaluated by using $i = m + k + 1$, where $m = 2^j$. the minimal value for which (3) holds is $i = 2(then\; m = 1, k = 0)$: the maximal value is $i = 2M$ where $M = 2^J$. The index$i = 1$ corresponds to the scaling function of the Haar wavelet

$$h_1(t) = \begin{cases} 1 & if\ t \in [0,1] \\ 0 & otherwise \end{cases}$$

Any function $f(t)$ in space $L^2([0,1])$ can be expanded into Haar series

$$f(t) = \sum_{i=0}^{\infty} a_i h_i(t) \tag{4}$$

Where a_i $, i = 0,1,2, \dots \dots \dots$ ....are the Haarcoefficients , which are given by

$$c_i = 2^j \int_0^1 f(t) h_i(t) dt \tag{5}$$

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Usually the series expansion of equation (4) contains infinite terms for a general smooth function $f(t)$, however if $f(t)$ is approximated as piecewise constant during each subinterval , equation (4) will be terminated at finite term

$$f(t) \cong \sum_{i=0}^{2M-1} a_i h_i(t) = A^T{}_{2M-1} H_{2M-1}(t) \tag{6}$$

The Haar coefficient vector is defined as

$$A_{2M-1} = [a_0, a_1, a_2, \dots, a_{2M-1}]^T \tag{7}$$

And the Haar coefficient vector $H_m(t)$ is defined as

$$H_{2M-1}(t) = [h_0, h_1, h_2, \dots, h_{2M-1}]^T \tag{8}$$

The collection points are taken as follows

$$t_\ell = \frac{\ell - 0.5}{2M} \tag{9}$$

Where $\ell = 1,2, \dots ,2M - 1$ and $H(i, \ell) = h_i(t_\ell)$ is the Haar coefficient matrix which is asquare matrix of the dimension $2M \times 2M$.

3.2 Operational matrix and its integrals:

The equation

$$P_{1,i}(t) = \int_0^t h_i(x)dx$$

Defines the operational matrix of integration, which is $2M \times 2M$ square matrix. The form

$$P_{s+1,i}(t) = \int_0^t P_{s,i}(x)dx$$

Where $i = 1,2, \ldots.,2M$ and $S = 1,2, \ldots, n$ is used to develop a general operational matrix . According to the function $P_{1,i}(t)$and using Eq (3) contracted the following integrals :

$$P_{1,i}(t) = \begin{cases} t - \frac{k}{m} & if\ t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{k+1}{m} - t & if\ t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & otherwise \end{cases} \tag{10}$$

$$P_{2,i}(t) = \begin{cases} \frac{1}{2!}\left(t - \frac{k}{m}\right)^2 & if\ t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{1}{4m^2} - \frac{1}{2!}\left(\frac{k+1}{m} - t\right)^2 & if\ t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ \frac{1}{4m^2} & if\ t \in \left[\frac{k+1}{m}, 1\right) \\ 0 & eleswhere \end{cases} \tag{11}$$

$$P_{3,i}(t) = \begin{cases} \frac{1}{6}\left(t - \frac{k}{m}\right)^3 & if\ t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ \frac{1}{4m^2}\left(t - \frac{k+0.5}{m}\right) - \frac{1}{6}\left(\frac{k+1}{m} - t\right)^3 & if\ t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ \frac{1}{4m^2}\left(t - \frac{k+0.5}{m}\right) & if\ t \in \left[\frac{k+1}{m}, 1\right) \\ 0 & eleswhere \end{cases} \tag{12}$$

And the generalized of Haar functions of order n are considered as

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$$P_{s,i}(t) = \begin{cases} \frac{1}{s!}\left(t-\frac{k}{m}\right)^{s} & if\ t \in \left[\frac{k}{m},\frac{k+0.5}{m}\right) \\ \frac{1}{s!}\left\{(t-\frac{k}{m})^{s}-2(t-\frac{k+0.5}{m})^{s}\right\} & if\ t \in \left[\frac{k+0.5}{m},\frac{k+1}{m}\right) \\ \frac{1}{s!}\left\{(t-\frac{k}{m})^{s}-2\left(t-\frac{k+0.5}{m}\right)^{s}+\left(t-\frac{k+1}{m}\right)^{s}\right\} & if\ t \in \left[\frac{k+1}{m},1\right) \\ 0 & eleswhere \end{cases} \tag{13}$$

These formulas hold for $i > 1$. if $i = 1$ we have

$$P_{s,1} = \frac{1}{s!}t^{s} \tag{14}$$

3.3. Haar wavelet method for solving differential equations "S. Pandit1 et.all (2014) "

If equation (1) is of the second order in the formula

$$\varepsilon \acute{\acute{u}} + \mu \acute{u} + cy = f(t) \qquad , t \in (0,1) \tag{15}$$

Subject to the initial condition

$$u(0) = \alpha \quad , \quad \acute{u}(0) = \beta$$

A simple and accurate Haar wavelet for problem (15) is constructed by approximating the highest – order derivative$\acute{\acute{u}}(t)$ using a Haar wavelet series as follows :

$$\acute{\acute{u}}(t) = \sum_{i=0}^{2M-1} a_i h_i(t) \tag{16}$$

On integration equation (16) we get $\acute{u}(t)$, $u(t)$ and finally $u(t)$ can be expanded in form of Haar wavelet series and its integrals.

$$\acute{u}(t) = \sum_{i=0}^{2M-1} a_i P_{1,i}(t) + \acute{u}(0) \quad (17)$$

$$u(t) = \sum_{i=0}^{2M-1} a_i \big(P_{2,i}(t) + \acute{u}(0)\big)t + u(0) \quad (18)$$

Were $P_{1,i}$, $P_{2,i}$ are defined in equations (10) , (11) .

Discretization using collocation points t_ℓ of equations (16)- (18) can be reduced into the following matrix form

$$\acute{\acute{u}} = \begin{bmatrix} h_0(t_0) & \cdots & h_{2M-1}(t_0) & 0 & 0 \\ \cdots & & \cdots & & \cdots \\ h_0(t_{2M-1}) & \cdots & h_{2M-1}(t_{2M-1}) & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (19)$$

$$\acute{u} = \begin{bmatrix} P_{1,0}(t_0) & \cdots & P_{1,2M-1}(t_0) & 1 & 0 \\ \vdots & & \vdots & & \vdots \\ P_{1,0}(t_{2M-1}) & \cdots & P_{1,2M-1}(t_{2M-1}) & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (20)$$

$$u = \begin{bmatrix} P_{2,0}(t_0) & \cdots & p_{2,2M-1}(t_0) & t_0 & 1 \\ \vdots & & \vdots & & \vdots \\ P_{2,0}(t_{2M-1}) & \cdots & P_{2,2M-1}(t_{2M-1}) & t_{2M-1} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (21)$$

The initial condition can be transformed into

$$\begin{bmatrix} u(0) \\ \acute{u}(0) \end{bmatrix} = \begin{bmatrix} P_{2,0}(0) & \cdots & p_{2,2M-1}(0) & 0 & 1 \\ P_{1,0}(0) & \cdots & p_{1,2M-1}(0) & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (22)$$

Where $a = [a_0, a_1, \dots, a_{2M-1}]^T$ $\quad and \quad$ $b = [\acute{u}(0)\,, u(0)]^T$

Now substitute the values of $\acute{\acute{u}}(t)$, $\acute{u}(t)$ and $u(t)$ in equation(15) and we obtain non homogeneous system of algebraic equations which contain $2M$ equations and $2M$ unknowns .

On solving this system, we get the Haar coefficients. The approximate solution can be determined using Haar coefficients , a_i , $i = 0,1, \dots ,2M - 1$ in equation (18) .

4. Numerical Experiment

Example 1 :

solve

$$\acute{\acute{u}} + 2\acute{u} + 5u = 3e^{-t} \sin t \tag{23}$$

Subject to

$$u(0) = 0 \quad , \quad \acute{u}(0) = 1$$

Exact solution of equation (23) is $u = e^{-t} \sin t$.

Let

$$\acute{\acute{u}}(t) = \sum_{i=0}^{2M-1} a_i h_i \tag{24}$$

Integrating equation (24) with respect initial conditions :

$$\acute{u}(t) = \sum_{i=0}^{2M-1} a_i P_{1,i}(t) + 1 \tag{25}$$

$$u(t) = \sum_{i=0}^{2M-1} a_i \big(P_{2,i}(t) + 1\big)t \tag{26}$$

Now substituting equations (24) , (25) , (26) in equation (23) , we get a system of algebraic equations .After solving it , we can obtain the Haar coefficients a_i. The numerical solution for $M = 8$ is shown in Table 1 and Figure 1.

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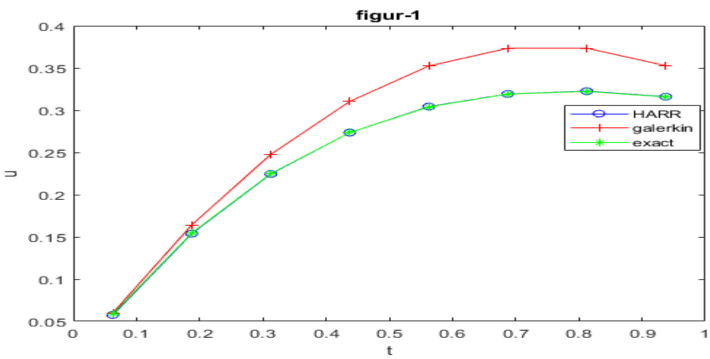
T	Exact solution	Haar solution	Absolute Error	Galerhin solution	Absolute Error
0.0625	0.0587	0.0576	0.0011	0.0599	**0.0012**
0.1875	0.1545	0.1539	0.0006	0.1642	**0.0097**
0.3125	0.2249	0.2245	0.0004	0.2477	**0.0228**
0.4375	0.2735	0.2735	0.0000	0.3104	**0.0369**
0.5625	0.3039	0.3042	0.0003	0.3523	**0.0484**
0.6875	0.3191	0.3192	0.0001	0.3733	**0.0542**
0.8125	0.3222	0.3225	0.0003	0.3734	**0.0512**
0.9375	0.3157	0.3160	0.0003	0.3527	**0.0370**

Table - 1



Exampel 2 :

Solve

$$t\acute{\acute{u}} + (1 - 2t)\acute{u} - 2u = 0 \qquad (27)$$

For the initial conditions

$$u(0) = 1 \quad , \quad \acute{u}(0) = 2$$

Exact solution of equation (27) is $u = e^{2t}$.

Let

$$\acute{u}(t) = \sum_{i=0}^{2M-1} a_i h_i(t) \tag{28}$$

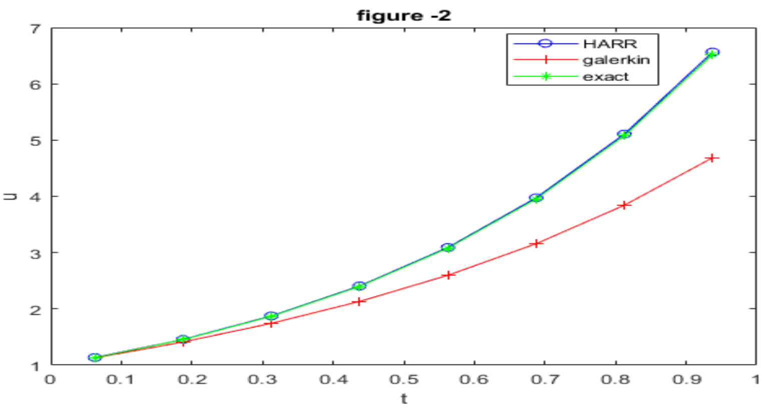
$$\therefore \quad \acute{u}(t) = \sum_{i=0}^{2M-1} a_i P_{1,i}(t) + 2 \tag{29}$$

$$\therefore \quad u(t) = \sum_{i=0}^{2M-1} a_i \big(P_{2,i}(t) + 2\big) t + 1 \tag{30}$$

Now substituting equations (28) , (29) , (30) in equation (27) , we get a system of algebraic equations . The numerical results of this example are shown in Table 2 and Figure 2.

T	Exact solution	Haar solution	Absolute Error	Galerhin solution	Absolute Error
0.0625	1.1331	1.1364	0.0033	1.1288	**0.0043**
0.1875	1.4550	1.4596	0.0046	1.4134	**0.0416**
0.3125	1.8682	1.8758	0.0076	1.7438	**0.1244**
0.4375	2.3989	2.4088	0.0099	2.1332	**0.2657**
0.5625	3.0802	3.0931	0.0129	2.5978	**0.4824**
0.6875	3.9551	3.9756	0.0205	3.1587	**0.7964**
0.8125	5.0784	5.1078	0.0294	3.8433	**1.2351**
0.9375	6.5208	6.5637	0.0429	4.6870	**1.8338**

Table 2



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Example 3:

Let us consider the differential equation

$$u^{(8)} - u(t) = -8e^{t} \tag{31}$$

Subjected to initial conditions

$$u(0) = 1\,, \acute{u}(0) = 0, \acute{\acute{u}}(0) = -1, \acute{\acute{\acute{u}}}(0) = -2, u^{(4)}(0) = -3$$

$$u^{(5)}(0) = -4\,, u^{(6)}(0) = -5\,\,, u^{(7)}(0) = -6$$

The Exact solution is $u(t) = (1 - t)e^{t}$.

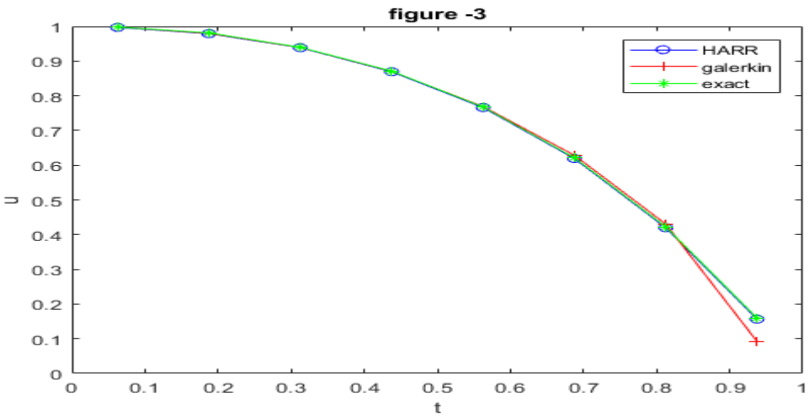
Let

$$u^{(8)}(t) = \sum_{i=0}^{2M-1} a_i h_i(t) \tag{32}$$

By successively integration equation (32) with respect to t from 0 to 1 and applying the initial conditions, substituting into equation (31) , and solving the resulting algebraic system , we obtain the numerical solution illustrated in Table 3 and Figure 3.

T	Exact solution	Haar solution	Absolute Error	Galerhin solution	Absolute Error
0.0625	0.9980	0.9970	0.0010	0.9980	**0.0000**
0.1875	0.9812	0.9790	0.0022	0.9800	**0.0012**
0.3125	0.9397	0.9386	0.0011	0.9394	**0.0003**
0.4375	0.8712	0.8693	0.0019	0.8707	**0.0005**
0.5625	0.7678	0.7661	0.0017	0.7690	**0.0012**
0.6875	0.6215	0.6193	0.0022	0.6289	**0.0074**
0.8125	0.4225	0.4200	0.0025	0.4305	**0.0080**
0.9375	0.1596	0.1565	0.0031	0.0921	**0.0675**

Table – 3



5. conclusion

This study compared the Haar wavelet and Galerkin methods for solving linear differential equations. It was found that the Haar wavelet method offers both higher accuracy and better computational efficiency, as indicated by the results shown in Tables 1,2 and3. Because the Galerkin method demands precise selection of basis functions capable of accurately representing the solution, contrasting with Haar wavelets, which offer a fixed and user-friendly basis.

Using MATLAB, numerical solutions to these equations were obtained.

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